Lecture Notes on Harmony

15-816: Substructural Logics
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Lecture 3
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In the last lecture we started developing a sequent calculus which expresses the notion of a *hypothetical judgment* $\Omega \vdash z$ where Ω is a ordered collection of antecendents and z is the succedent. Each connective (we discussed $x \,/\, y$, $x \setminus y$, $x \bullet y$, $\mathbf{1}$ and $x \mathbin{\&} y$) was defined by a right rule (which shows how to prove such a proposition) and a left rule (which shows how to use such a proposition).

We wrote down rules that seemed intuitively correct, but we were surprised we couldn't prove $(x \bullet y) \setminus z \vdash y \setminus (x \setminus z)$ with our rules. We need some criteria to decide if the rules are "correct". In traditional development of classical logic this is usually done by developing a Tarskian semantics, interpreting the propositions in some mathematical domain, and then assessing whether the rules are sound and complete with respect to this interpretation.

An alternative approach views the left and right rules of the sequent calculus themselves as providing the meaning of the connectives, a so-called *proof-theoretic semantics*. This idea was pioneered by Gentzen [Gen35], developed further by Dummett [Dum91], and proposed as the foundation of type theory by Martin-Löf [ML83]. We follow this approach here in the definition of substructural logic and the sequent calculus, rather than intuitionistic logic and natural deduction.

We provide two tests to verify if the left and right rules for a connective are in *harmony*, which permits us to view the rules as a semantic definition. These criteria not only supply *internal* notions of soundness and completeness, but they will also play a critical role later on, when we introduce computational interpretations of proofs.

1 Repairing the Rules for $y \setminus x$

Recall the relevant preliminary rules from Lecture 2.

$$\frac{}{x \vdash x}\ \mathsf{id}_x$$

$$\frac{y\ \Omega \vdash x}{\Omega \vdash y \setminus x}\ \setminus R \qquad \frac{\Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ y\ (y \setminus x)\ \Omega_R \vdash z}\ \setminus L^*$$

$$\frac{\Omega \vdash x \quad \Omega' \vdash y}{\Omega\ \Omega' \vdash x \bullet y}\ \bullet R \qquad \frac{\Omega_L\ x\ y\ \Omega_R \vdash z}{\Omega_L\ (x \bullet y)\ \Omega_R \vdash z}\ \bullet L$$

Read from the conclusion to the premise, each of the right and left rules removes a connective from the sequent, so they analyze the structure of the proposition in the endsequent.

When we were trying to prove $(x \bullet y) \setminus z \vdash y \setminus (x \setminus z)$ we got stuck at the following point, with no further rule applicable. As usual in the sequent calculus, you should read this partial derivation from the bottom upwards.

$$\dfrac{\dfrac{\overset{?}{x \quad y \quad (x \bullet y) \setminus z \vdash z}}{y \quad (x \bullet y) \setminus z \vdash x \setminus z}\ \setminus R}{(x \bullet y) \setminus z \vdash y \setminus (x \setminus z)}\ \setminus R$$

At this point, there are a couple of directions we could go in. One is to add new rules, another one is to generalize the rules we already have. One suggestion would be to add the rule

$$\frac{\Omega_L\ (x \bullet y)\ \Omega_R \vdash z}{\Omega_L\ x\ y\ \Omega_R \vdash z}$$

This would allow us to complete this proof because we could combine x and y so that $\setminus L^*$ then applies. This is a valid direction to consider (and there are solutions along these lines in other contexts), but a calculus with this rule no longer breaks down connectives as we go up the proof. If we see the rules as a semantic definition, they would now be problematic since the meaning would no longer be compositional but may depend on other propositions that we did not anticipate. We also need to modify our simple argument for decidability. Finally, it seems like a very special case: how do we know we have added enough rules?

A second approach is to generalize the rule

$$\dfrac{\Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ y\ (y \setminus x)\ \Omega_R \vdash z}\ {\setminus}L^*$$

Instead of requiring the proposition to the left of $(y \setminus x)$ to match y *exactly*, all we need is that we can *prove* y from some of the antecedents. Since order is important, we slice off a section just to the left of $(y \setminus x)$ for this purpose to arrive at the following rule.

$$\dfrac{\Omega' \vdash y \quad \Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ \Omega'\ (y \setminus x)\ \Omega_R \vdash z}\ {\setminus}L$$

A pleasing aspect of this rule is that it just breaks down a single connective, so it fits within our general program of right and left rules and meaning explanations of propositions. We will therefore adopt it. Spoiler alert: it will pass the tests for harmony we devise in the next section. Moreover, the previous rule ${\setminus}L^*$ can easily be justified as a *derived rule of inference*:

$$\dfrac{\dfrac{}{y \vdash y}\ \mathsf{id}_y \qquad \Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ y\ (y \setminus x)\ \Omega_R \vdash z}\ {\setminus}L$$

Finally, the rule is strong enough so we can complete the proof in our motivating example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x \vdash x}\ \mathsf{id}_x \quad \dfrac{}{y \vdash y}\ \mathsf{id}_y}{x \quad y \vdash x \bullet y}\ {\bullet}R \quad \dfrac{}{z \vdash z}\ \mathsf{id}_z}{x \quad y \quad (x \bullet y) \setminus z \vdash z}\ {\setminus}L}{y \quad (x \bullet y) \setminus z \vdash x \setminus z}\ {\setminus}R}{(x \bullet y) \setminus z \vdash y \setminus (x \setminus z)}\ {\setminus}R$$

2 Identity and Cut

The key idea behind the tests we devise on the harmony between left and right rules is that they ensure agreement between *proving a proposition* (the right rule) and *using a proposition* (the left rule). How can we embody these principles in the sequent calculus? Actually, we already have one rule that embodies the second one, name the identity rule.

$$\frac{}{x \vdash x}\ \mathsf{id}_x$$

It expresses that if we have an antecedent x (and nothing else) we can use it to prove the succedent x.

The other direction would say that if we can *prove* x we are justified to *use* x. In its simplest form, this would be

$$\frac{\vdash x \quad x \vdash z}{\vdash z}$$

This is too restrictive, since we should be able to use x even if it requires some of the antecedents in the conclusion. Ordering constraints mean what is used to prove x should be some segment of our context.

$$\frac{\Omega' \vdash x \quad \Omega_L\ x\ \Omega_r \vdash z}{\Omega_L\ \Omega'\ \Omega_R \vdash z}\ \mathsf{cut}_x$$

If, for the whole logical system, the left and right rules are in balance, we should never need identity or cut. That's actually not quite true: if we have propositional variables like x or y because we perform schematic inference (as we have been doing in much of the development), then identity for variables cannot be eliminated. These two properties are known as *identity elimination* and *cut elimination*.

Unfortunately, cut and identity elimination are global properties of a complete logical system, not isolated questions about the individual left and right rules. In the next lecture we will proceed to prove, as metatheorems about the Lambek calculus, that these two properties actually hold. In this lecture we will focus on isolating local transformations on proofs which we call *identity expansion* and *cut reduction* which are isolated checks on the left and right rules for each connective. Cut reduction in particular will also play a fundamental role in the computational interpretation of proofs we will discuss later in the course.

Consider a proposition $x * y$ for some connective $*$. We say it satisfies *identity expansion* if we can replace the identity at $x * y$ by uses of identities at x and y. Conversely, we say it satisfies *cut reduction* if we can replace any cut at $x * y$ that matches a right rule for $x * y$ against the left rule on the same proposition by cuts at x and y. In the next few sections we will check these property for some connectives and we will also look for counterexamples to understand what may happen if these properties are not satisfied.

3 Harmony for $y \setminus x$

We'll start with $y \setminus x$. Identity expansion is easier to check. Can we *expand* the one-line proof

$$\frac{}{y \setminus x \vdash y \setminus x}\ \mathsf{id}_{y\setminus x}$$

into a proof just using identity at x and y? There are only two possible rules that could apply, but $\setminus L$ will fail since we cannot prove y. So wee need to start with $\setminus R$.

$$\frac{\begin{array}{c}\vdots \\ y \quad y \setminus x \vdash x\end{array}}{y \setminus x \vdash y \setminus x}\ \setminus R$$

Fortunately, at this point we can use $\setminus L$ followed by identities to complete the proof.

$$\frac{\dfrac{\dfrac{}{y \vdash y}\ \mathsf{id}_y \quad \dfrac{}{x \vdash x}\ \mathsf{id}_x}{y \quad y \setminus x \vdash x}\ \setminus L}{y \setminus x \vdash y \setminus x}\ \setminus R$$

Interesting, the restricted rule $\setminus L^*$ would have actually passed this test. It might have been slightly suspicious, though, because the expanded form does not need id_y.

Note that the expansion *introduces* uses of $\setminus R$ and $\setminus L$ into the proof. Cut reduction instead *eliminates* uses of these rules that appear just above the cut. The situation:

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ y\ \Omega \vdash x\end{array}}{\Omega \vdash y \setminus x}\ \setminus R \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega' \vdash y\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ \Omega'\ (y \setminus x)\ \Omega_R \vdash z}\ \setminus L}{\Omega_L\ \Omega'\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_{y\setminus x}$$

We have named here the subproofs, $\mathcal{D}$, $\mathcal{E}'$, and $\mathcal{E}$, since these are the proofs we can now use to justify the conclusion, using cut only at y and x. Indeed, we can first cut $\mathcal{E}'$ with $\mathcal{D}$ and then the result with $\mathcal{E}$. Note that we could also cut $\mathcal{D}$ with $\mathcal{E}$ and then the result with $\mathcal{E}'$. We show the first alternative.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega' \vdash y\end{array} \quad \begin{array}{c}\mathcal{D}\\ y\ \Omega \vdash x\end{array}}{\Omega'\ \Omega \vdash x}\ \mathsf{cut}_y \qquad \begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ \Omega'\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_x$$

At this point we know we have established harmony. To summarize, we have the identity expansion, denoted by $\Longrightarrow_E$

$$\dfrac{}{y \setminus x \vdash y \setminus x}\ \mathsf{id}_{y\setminus x} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\dfrac{}{y \vdash y}\ \mathsf{id}_y \quad \dfrac{}{x \vdash x}\ \mathsf{id}_x}{y \quad y \setminus x \vdash x}\ {\setminus}L}{y \setminus x \vdash y \setminus x}\ {\setminus}R$$

and the cut reduction, denoted by $\Longrightarrow_R$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ y\ \Omega \vdash x\end{array}}{\Omega \vdash y \setminus x}\ {\setminus}R \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega' \vdash y\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ \Omega'\ (y \setminus x)\ \Omega_R \vdash z}\ {\setminus}L}{\Omega_L\ \Omega'\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_{y\setminus x}$$

$$\Longrightarrow_R \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega' \vdash y\end{array} \quad \begin{array}{c}\mathcal{D}\\ y\ \Omega \vdash x\end{array}}{\Omega'\ \Omega \vdash x}\ \mathsf{cut}_y \qquad \begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ \Omega'\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_x$$

Let's see what would happen if we had the weaker ${\setminus}L^*$ rule.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ y\ \Omega \vdash x\end{array}}{\Omega \vdash y \setminus x}\ {\setminus}R \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ y\ (y \setminus x)\ \Omega_R \vdash z}\ {\setminus}L^*}{\Omega_L\ y\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_{y\setminus x}$$

$$\Longrightarrow_R \quad \dfrac{\begin{array}{c}\mathcal{D}\\ y\ \Omega \vdash x\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ y\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_x$$

We note that, perhaps surprisingly, cut reduction can still apply, but that the situation of cut we are considering is not most general, that is, it only applies in the special case of a cut where y happens to be present in its second premise. This would eventually lead to a failure of the global cut elimination property we discuss in the next lecture.

Coming back to the original rules, let's make another (not completely implausible) mistake and consider the incorrect $\backslash R^?$ rule which adds y to the wrong side of the antecedents.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Omega\ y \vdash x\end{matrix}}{\Omega \vdash y \backslash x}\,\backslash R^? \qquad \dfrac{\begin{matrix}\mathcal{E}'\\ \Omega' \vdash y\end{matrix} \quad \begin{matrix}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{matrix}}{\Omega_L\ \Omega'\ (y \backslash x)\ \Omega_R \vdash z}\,\backslash L}{\Omega_L\ \Omega'\ \Omega\ \Omega_R \vdash z}\,\mathsf{cut}_{y\backslash x}$$

Now we can try to perform a similar reduction to before, which would give us:

$$\Longrightarrow^?_R \quad \dfrac{\dfrac{\begin{matrix}\mathcal{E}'\\ \Omega' \vdash y\end{matrix} \quad \begin{matrix}\mathcal{D}\\ \Omega\ y \vdash x\end{matrix}}{\Omega\ \Omega' \vdash x}\,\mathsf{cut}_y \qquad \begin{matrix}\mathcal{E}\\ \Omega_L\ x\ \Omega_R \vdash z\end{matrix}}{\Omega_L\ \Omega\ \Omega'\ \Omega_R \vdash z}\,\mathsf{cut}_x$$

We note that this proof has a different conclusion from before, swapping Ω' and Ω, so this is not a valid reduction. Indeed, cut reduction fails: we cannot construct, from the proofs $\mathcal{D}$, $\mathcal{E}'$ and $\mathcal{E}$ that we have a proof of the original endsequent $\Omega_L\ \Omega'\ \Omega\ \Omega_R$ using cut only at x and y.

How catastrophic is this failure? See Exercise 2. Suffice it to say here that our test fails. The left and right rules are not in harmony. Indeed, the identity expansion would fail as well:

$$\dfrac{}{y \backslash x \vdash y \backslash x}\,\mathsf{id}_{y\backslash x} \quad \Longrightarrow^?_E \quad \dfrac{\dfrac{\begin{matrix}??\\ \vdash y\end{matrix} \quad \begin{matrix}??\\ x\ y \vdash x\end{matrix}}{(y \backslash x)\ y \vdash x}\,\backslash L}{y \backslash x \vdash y \backslash x}\,\backslash R^?$$

Each step here is forced, and we can not prove either of the two sequents at the top since no rule applies.

We skip harmony for x / y, which is symmetric (see Exercise 1) and move on to other connectives.

4 Alternative Conjunction

As one more example, let's consider the alternative conjunction $x \& y$. Recall the sequent rules

$$\frac{\Omega \vdash x \quad \Omega \vdash y}{\Omega \vdash x \mathbin{\&} y}\ \&R \qquad \frac{\Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ x \mathbin{\&} y\ \Omega_R \vdash z}\ \&L_1 \quad \frac{\Omega_L\ y\ \Omega_R \vdash z}{\Omega_L\ x \mathbin{\&} y\ \Omega_R \vdash z}\ \&L_2$$

The identity expansion is once again straightforward, since essentially every step is forced.

$$\frac{}{x \mathbin{\&} y \vdash x \mathbin{\&} y}\ \mathsf{id}_{x \& y} \quad \Longrightarrow_E \quad \frac{\dfrac{\dfrac{}{x \vdash x}\ \mathsf{id}_x}{x \mathbin{\&} y \vdash x}\ \&L_1 \quad \dfrac{\dfrac{}{y \vdash y}\ \mathsf{id}_y}{x \mathbin{\&} y \vdash y}\ \&L_2}{x \mathbin{\&} y \vdash x \mathbin{\&} y}\ \&R$$

What would happen if, say, we forgot the second left rule, $\&L_2$? We would not be able to complete the proof of the second premise of $\&R$, so the identity expansion would fail. It is perhaps easier to see here than in the case of $x \setminus y$, that identity expansion verifies that, taken together, the left rules for a connective are strong enough to prove the succedent with this same connective. If we omit the second left rule here they are too weak and our test will fail.

For the cut reduction, we actually have to test two situations, since there are two possible left rules to infer $x \mathbin{\&} y$. First, with the first left rule:

$$\frac{\dfrac{\overset{\mathcal{D}}{\Omega \vdash x} \quad \overset{\mathcal{D}'}{\Omega \vdash y}}{\Omega \vdash x \mathbin{\&} y}\ \&R \quad \dfrac{\overset{\mathcal{E}}{\Omega_L\ x\ \Omega_R \vdash z}}{\Omega_L\ (x \mathbin{\&} y)\ \Omega_R \vdash z}\ \&L_1}{\Omega_L\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_{x \& y}$$

We can easily reduce this to a cut between $\mathcal{D}$ and $\mathcal{E}$.

$$\Longrightarrow_R \quad \frac{\overset{\mathcal{D}}{\Omega \vdash x} \quad \overset{\mathcal{E}}{\Omega_L\ x\ \Omega_R \vdash z}}{\Omega_L\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_x$$

We don't need a cut on y here: the proof of $x \mathbin{\&} y$ offers a choice between the proof of x and the proof of y and in this case the proof that uses $x \mathbin{\&} y$ chooses x.

Of course, if the second premise of the cut is $\& L_2$, we perform a symmetric reduction, this time to a cut on y. We omit the straightforward deduction.

Note that if we had mistakenly omitted the $\& L_2$ rule, then we would have had only the first case to check, and it would pass. In other words, the left rules are not too strong. In this case, the imbalance can only be noted in the identity expansion.

Another suggested mode of failure would require two copies of x in the left rule. Then the situation would be as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Omega \vdash x\end{array} \quad \begin{array}{c}\mathcal{D}'\\ \Omega \vdash y\end{array}}{\Omega \vdash x \mathbin{\&} y}\,\& R \qquad \dfrac{\begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ (x \mathbin{\&} y)\ \Omega_R \vdash z}\,\& L_1^?}{\Omega_L\ \Omega\ \Omega_R \vdash z}\,\mathsf{cut}_{x \& y}$$

A cut reduction would require two cuts on x to eliminate the two copies, but this would duplicate Ω and lead to the wrong endsequent.

$$\Longrightarrow_R^? \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Omega \vdash x\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Omega \vdash x\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Omega_L\ x\ x\ \Omega_R \vdash z\end{array}}{\Omega_L\ \Omega\ x\ \Omega_R \vdash z}\,\mathsf{cut}_x}{\Omega_L\ \Omega\ \Omega\ \Omega_R \vdash z}\,\mathsf{cut}_x$$

So this is *not* a valid cut reduction. The right rule and this modified left rule would not be in harmony. In fact, identity expansion would also fail since we have an extra copy of x in one branch on the proof, which is not allowed in applications of the identity rule.

5 Rule Summary

Here is a summary of the sequent calculus rules for the Lambek calculus so far [Lam58].[1] We often consider the *cut-free* sequent calculus, omitting the cut_x rule, and the *identity expanded* sequent calculus, restricting the id_x rule to propositional variables x.

We refer to id_x and cut_x as *judgmental rules* since they are concerned only with the nature of the ordered hypothetical judgment but not any particular propositions. They are also sometimes called *structural rules*. The other

[1] Actually, Lambek did not have **1** or $\&$ as explicit connectives.

rules, namely the right and left rules, are defining propositional connectives so we call them the *propositional rules*.

Judgmental rules

$$\frac{}{x \vdash x}\ \mathsf{id}_x \qquad \frac{\Omega \vdash x \quad \Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ \Omega\ \Omega_R \vdash z}\ \mathsf{cut}_x$$

Propositional rules

$$\frac{y\ \Omega \vdash x}{\Omega \vdash y \setminus x}\ \setminus R \qquad \frac{\Omega' \vdash y \quad \Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ \Omega'\ (y \setminus x)\ \Omega_R \vdash z}\ \setminus L$$

$$\frac{\Omega\ y \vdash x}{\Omega \vdash x \,/\, y}\ /R \qquad \frac{\Omega' \vdash y \quad \Omega_L\ x\ \Omega_r \vdash z}{\Omega_L\ (x \,/\, y)\ \Omega'\ \Omega_R \vdash z}\ /L$$

$$\frac{\Omega \vdash x \quad \Omega' \vdash y}{\Omega\ \Omega' \vdash x \bullet y}\ \bullet R \qquad \frac{\Omega_L\ x\ y\ \Omega_R \vdash z}{\Omega_L\ (x \bullet y)\ \Omega_R \vdash z}\ \bullet L$$

$$\frac{}{\vdash \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Omega_L\ \Omega_R \vdash z}{\Omega_L\ \mathbf{1}\ \Omega_R \vdash z}\ \mathbf{1}L$$

$$\frac{\Omega \vdash x \quad \Omega \vdash y}{\Omega \vdash x \,\&\, y}\ \& R \qquad \frac{\Omega_L\ x\ \Omega_R \vdash z}{\Omega_L\ x \,\&\, y\ \Omega_R \vdash z}\ \& L_1 \quad \frac{\Omega_L\ y\ \Omega_R \vdash z}{\Omega_L\ x \,\&\, y\ \Omega_R \vdash z}\ \& L_2$$

Exercises

Exercise 1 Show identity expansion and cut reduction for $x \mathbin{/} y$.

Exercise 2 As pointed in Section 3, if we replace the $\backslash R$ rule with $\backslash R^?$ which adds y at the wrong end of the antecedents, both identity expansion and cut reduction fail. Explore which, if any, of the three structural rules would now be derivable in the presence of cut and identity. In each case, show the proof if one exists or indicate that you believe it is not derivable (which you do not need to prove).

1. Exchange:
$$\frac{\Omega_L \; y \; x \; \Omega_R \vdash z}{\Omega_L \; x \; y \; \Omega_R \vdash z}\ \textsf{exchange}$$

2. Weakening:
$$\frac{\Omega_L \; \Omega_R \vdash z}{\Omega_L \; x \; \Omega_R \vdash z}\ \textsf{weaken}$$

3. Contraction:
$$\frac{\Omega_L \; x \; x \; \Omega_R \vdash z}{\Omega_L \; x \; \Omega_R \vdash z}\ \textsf{contract}$$

If any of these were derivable, it would quantify the global effect of the lack of harmony for a single connective, upsetting our intended meaning of the logic. If none of these can you find another expression of the failure of semantic intent?

Exercise 3 Assume we define a new connective $*$ with the following right and left rules (which mix the right rule for alternative conjunctions with left rule for concatenation):

$$\frac{\Omega \vdash x \quad \Omega \vdash y}{\Omega \vdash x * y}\ {*R} \qquad \frac{\Omega_L \; x \; y \; \Omega_R \vdash z}{\Omega_L \; (x * y) \; \Omega_R \vdash z}\ {*L}$$

First, show which of identity expansion and cut reduction fail and which succeed. Then answer the same questions as in Exercise 2.

Exercise 4 Assume we define a new connective $\#$ with the following right and left rules (which mix the right rule for concatenation with left rules for alternative conjunction):

$$\frac{\Omega_L \vdash x \quad \Omega_R \vdash y}{\Omega_L \; \Omega_R \vdash x \mathbin{\#} y}\ \#R \quad \frac{\Omega_L \; x \; \Omega_R \vdash z}{\Omega_L \; (x \mathbin{\#} y) \; \Omega_R \vdash z}\ \#L_1 \quad \frac{\Omega_L \; y \; \Omega_R \vdash z}{\Omega_L \; (x \mathbin{\#} y) \; \Omega_R \vdash z}\ \#L_2$$

First, show which of identity expansion and cut reduction fail and which succeed. Answer the same questions as in Exercise 2.

Exercise 5 Consider a connective $x \circ y$ (pronounced x *twist* y) defined in the original style of Lambek by

$$\frac{x \circ y}{y \quad x}\ \mathsf{twist}$$

Investigate this connective by going through the following steps.

1. Define the right and left rules for $x \circ y$.
2. Verify identity expansion and cut reduction for $x \circ y$.
3. Prove or refute that $x \circ y \vdash y \bullet x$ and $y \bullet x \vdash x \circ y$.
4. Find a curried equivalent $A(x, y, z)$ of $(x \circ y) \setminus z$ and prove $A(x, y, z) \vdash (x \circ y) \setminus z$ and $(x \circ y) \setminus z \vdash A(x, y, z)$.
5. Find a curried equivalent $B(x, y, z)$ of $x \,/\, (y \circ z)$ and prove $B(x, y, z) \vdash x \,/\, (y \circ z)$ and $x \,/\, (y \circ z) \vdash B(x, y, z)$.

Exercise 6 Consider propositions in the Lambek calculus constructed from $x \,/\, y$, $x \setminus y$, $x \bullet y$, $x \circ y$, and $\mathbf{1}$. This calculus should have some strong symmetries. Find a transformation $\overline{x}$ such that $\vdash x$ if and only if $\vdash \overline{x}$ that exhibits such a symmetry and prove that it satisfies this property.

Exercise 7 We have explained logical equivalence between x and y as $x \vdash y$ and $y \vdash x$. Can we internalize logical equivalence as a connective $x \equiv y$? Its defining rules in Lambek's original style would be

$$\frac{x \quad x \equiv y}{y}\ \mathsf{equiv}_1 \qquad \frac{x \equiv y \quad y}{x}\ \mathsf{equiv}_2$$

Answer the following questions if you find this is a proper connective, or explain if no satisfactory rules seem possible.

1. Define right and left rules for $x \equiv y$.
2. Verify identity expansion and cut reduction for $x \equiv y$.
3. Prove or refute that $\equiv$ is reflexive, symmetric, and transitive.
4. If you can define $x \equiv y$ *notationally* as proposition $A(x, y)$ with connectives already present rather than by right and left rules, show that $A \vdash x \equiv y$ and $x \equiv y \vdash A$ with your rules from part 1.

Exercise 8 Prove that $x \bullet \mathbf{1}$ and $\mathbf{1} \bullet x$ are equivalent to x.

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