Lecture Notes on Classical Linear Logic

15-816: Substructural Logics
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Lecture 25
December 8, 2016

Originally, linear logic was conceived by Girard [Gir87] as a *classical* system, with one-sided sequents, an involutive negation, and an appropriate law of excluded middle. For a number of the applications, such as functional computation, logic programming, and implicit computational complexity the intuitionistic version is more suitable. In the case of concurrent computation, both classical and intuitionistic systems may be used, although the additional expressiveness afforded by the intuitionistic system seems to have some advantages even in that setting.

In this lecture we present classical linear logic and then show that we can easily interpret it intuitionistically. Briefly, classical linear logic can be modeled intuitionistically as deriving a contradiction from linear assumptions. This is shown via a so-called *double-negation translation*. Its parametric nature allows a number of additional variants of classical linear logic to be explained intuitionistically, in particular the so-called *mix* rules.

These lecture notes do *not* present the operational semantics of classical linear logic as a basis for concurrency which we presented in lecture. The one we presented can be found in Section 5 of [CPT16] in a somewhat different notation, another semantics is given by Wadler [Wad12].

1 Classical Linear Sequents

A sequent in classical linear logic just has the form

$$\vdash A_1, \ldots, A_n$$

where $A_1, \ldots, A_n$ are propositions. The comma separator can be read as a form of disjunction, which does not exist in intuitionistic linear logic.

An important aspect of the system is a negation operator, written as $A^\perp$, which is defined for all propositions except that atomic ones. As we define the rules for the classical connectives, we will also have to define negation. We already note that

$$(A^\perp)^\perp = A$$

is one of the basic laws.

The identity emphasizes the fact that reasoning in classical linear logic is akin to deriving a contradiction.

$$\frac{}{\vdash A, A^\perp}\ \mathsf{id}$$

Cut is somehow the dual—we do not cut a proposition, but a proposition and its negation.

$$\frac{\vdash \Sigma, A \quad \vdash \Sigma', A^\perp}{\vdash \Sigma, \Sigma'}\ \mathsf{cut}$$

2 Multiplicative Connectives

The connectives now are no longer defined by left and right rules, but by right rules, negation, and right rules for the negated proposition. We can see that this must be the case by looking at the cut rule.

The multiplicative conjunction $A \otimes B$ is quite similar to the intuitionistic version.

$$\frac{\vdash \Sigma, A \quad \vdash \Sigma', B}{\vdash \Sigma, \Sigma', A \otimes B}\ \otimes$$

The negation $(A \otimes B)^\perp = A^\perp \mathbin{⅋} B^\perp$ introduces a new connective $⅋$ which does not exist in intuitionistic linear logic.

$$\frac{\vdash \Sigma, A, B}{\vdash \Sigma, A \mathbin{⅋} B}\ ⅋$$

It is a multiplicative form of disjunction, and clearly satisfies the law of excluded middle $A \mathbin{⅋} A^\perp$. We can check the cut reduction and identity expansion, just as we did in the intuitionistic case. First, the cut reduction:

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash \Sigma', B}{\vdash \Sigma, \Sigma', A \otimes B}\otimes \quad \dfrac{\vdash \Sigma'', A^\perp, B^\perp}{\vdash \Sigma'', A^\perp \mathbin{⅋} B^\perp}⅋}{\vdash \Sigma, \Sigma', \Sigma''}\mathsf{cut}$$

$$\longrightarrow_R$$

$$\dfrac{\vdash \Sigma', B \quad \dfrac{\vdash \Sigma, A \quad \vdash \Sigma'', A^\perp, B^\perp}{\vdash \Sigma, \Sigma'', B^\perp}\mathsf{cut}}{\vdash \Sigma, \Sigma', \Sigma''}\mathsf{cut}$$

Second, the identity expansion:

$$\dfrac{}{\vdash A \otimes B, A^\perp \mathbin{⅋} B^\perp}\mathsf{id}_{A\otimes B} \quad \longrightarrow_R \quad \dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\mathsf{id}_A \quad \dfrac{}{\vdash B, B^\perp}\mathsf{id}_B}{\vdash A \otimes B, A^\perp, B^\perp}\otimes}{\vdash A \otimes B, A^\perp \mathbin{⅋} B^\perp}⅋$$

We will not continue to do so, but leave it as an exercise to check cut reduction and identity expansion.

The multiplicative units do not present surprises. Note that unlike $A \mathbin{⅋} B$, $\perp$ can actually be given meaning intuitionistically.

$$\dfrac{}{\vdash \mathbf{1}}\mathbf{1} \qquad \mathbf{1}^\perp = \perp \qquad \dfrac{\vdash \Sigma}{\vdash \Sigma, \perp}\perp$$

Not surprisingly, $\perp$ is the identity for $⅋$.

3 Additive Connectives

The additives do not differ much in their intuitionistic and classical versions.

$$\dfrac{\vdash \Sigma, A}{\vdash \Sigma, A \oplus B}\oplus_1 \qquad \dfrac{\vdash \Sigma, B}{\vdash \Sigma, A \oplus B}\oplus_2$$

In a classical calculus, $\oplus$ and $\&$ are duals

$$(A \oplus B)^\perp = A^\perp \mathbin{\&} B^\perp$$

and the rule for $\&$ are as expected, copying the context Σ to both premises.

$$\frac{\vdash \Sigma, A \quad \vdash \Sigma, B}{\vdash \Sigma, A \& B} \&$$

The units present no particular surprises or difficulties.

$$\frac{}{\vdash \Sigma, \top} \top \qquad \top^{\perp} = \mathbf{0} \qquad \text{no rule for } \mathbf{0}$$

4 Exponential Modalities

Girard's formulation of the modalities was in terms of explicit rules for weakening, contraction, and dereliction. However, it is also possible to present classical linear logic using two judgments, truth and possibility. This is what Andreoli [And92] calls the *dyadic* formulation of linear logic. We show here the original rules for reference; other two-sided formulations can be found in [CCP03]. Note that Girard's formulation does not lend itself to a structural proof of cut elimination, which Andreoli did not present but can be found in [CCP03] and goes back to an another unpublished technical report [Pfe94].

In order to explain the rules for $!A$ we have to define its dual,

$$(!A)^{\perp} = ?A^{\perp}$$

Persistent resources become formulas $?A$, because we are working just on the right of the sequent. The ! rule requires there to be no linear resources, but permits persistent ones. These are now marked with $?$, so we obtain

$$\frac{\vdash ?\Sigma, A}{\vdash ?\Sigma, !A} \, !$$

Conversely, a persistent formula is true, which becomes

$$\frac{\vdash \Sigma, A}{\vdash \Sigma, ?A} \, ?$$

Why do not retain a copy of $?A$ in the premise, because we have explicit rules for weakening and contraction of persistent propositions.

$$\frac{\vdash \Sigma}{\vdash \Sigma, ?A} \text{Weaken} \qquad \frac{\vdash \Sigma, ?A, ?A}{\vdash \Sigma, ?A} \text{Contract}$$

5 Double-Negation Interpretation

We now follow [CCP03], interpreting classical linear logic in intuitionistic linear logic. The technique of a *double-negation translation* is quite common in logics [Fri78] and is related to conversion to continuation-passing style in programming languages.

Roughly, we think of classical $\vdash \Sigma$ as intuitionistic $\neg[\![\Sigma]\!] \vdash \bot$, that is, deriving a contradiction from the negation of the translation of Σ. It is not immediately clear what should play the role of negation on the intuitionistic side, however. Instead of using $\bot$ and $\neg A$ (which we have yet to define intuitionistically), we use a new atomic proposition p and translate $\vdash \Sigma$ to $[\![\Sigma]\!]_p \multimap p \vdash p$. We will later exploit the fact that the translation is parametric in p by considering some choices for what p might be. We write

$$\sim_p A = A \multimap p$$

to emphasize the interpretation of the translation as a form of negation. We usually omit the p, since it is never changed throughout a translation.

The theorem we are striving for is

$$\vdash \Sigma \quad \text{iff} \quad \sim_p [\![\Sigma]\!]_p \vdash p$$

Instead of just presenting the translation, we consider various cases to see what it should be. For example, what happens with atoms? Could we just translate atoms to themselves?

$$\frac{}{\vdash P, P^\perp}\ \mathsf{id}$$

If we set

$$\begin{array}{lcl} [\![P]\!] & = & P \\ [\![P^\perp]\!] & = & \sim P \end{array}$$

This means we would have to prove

$$\sim P, \sim(\sim P) \vdash p$$

which is

$$P \multimap p, (P \multimap p) \multimap p \vdash p$$

and easy to show.

Let's try $A \otimes B$.

$$\frac{\vdash \Sigma, A \quad \vdash \Sigma', B}{\vdash \Sigma, \Sigma', A \otimes B}\ \otimes$$

If we generate a tensor, but double-negate the subformulas,

$$[\![A \otimes B]\!] = ({\sim}{\sim}[\![A]\!]) \otimes ({\sim}{\sim}[\![B]\!])$$

then the sequent we have to show after translation would be

$${\sim}[\![\Sigma]\!], {\sim}[\![\Sigma']\!], {\sim}(({\sim}{\sim}[\![A]\!]) \otimes ({\sim}{\sim}[\![B]\!])) \vdash p$$

After applying $\multimap L$ and closing the subgoals with the identity, we are looking at

$$\dfrac{\dfrac{\vdots}{{\sim}[\![\Sigma]\!], {\sim}[\![\Sigma']\!] \vdash ({\sim}{\sim}[\![A]\!]) \otimes ({\sim}{\sim}[\![B]\!])} \quad \dfrac{}{p \vdash p}\,\mathsf{id}}{{\sim}[\![\Sigma]\!], {\sim}[\![\Sigma']\!], {\sim}(({\sim}{\sim}[\![A]\!]) \otimes ({\sim}{\sim}[\![B]\!])) \vdash p}\multimap L$$

Now we can apply the $\otimes R$ rule and then $\multimap R$ to bring ${\sim}[\![A]\!]$ back to the left-hand side.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{{\sim}[\![\Sigma]\!], {\sim}[\![A]\!] \vdash p}}{{\sim}[\![\Sigma]\!] \vdash {\sim}{\sim}[\![A]\!]}\multimap R \quad \dfrac{\dfrac{\vdots}{{\sim}[\![\Sigma']\!], {\sim}[\![B]\!] \vdash p}}{{\sim}[\![\Sigma']\!] \vdash {\sim}{\sim}[\![B]\!]}\multimap R}{{\sim}[\![\Sigma]\!], {\sim}[\![\Sigma']\!] \vdash ({\sim}{\sim}[\![A]\!]) \otimes ({\sim}{\sim}[\![B]\!])}\otimes R \quad \dfrac{}{p \vdash p}\,\mathsf{id}}{{\sim}[\![\Sigma]\!], {\sim}[\![\Sigma']\!], {\sim}(({\sim}{\sim}[\![A]\!]) \otimes ({\sim}{\sim}[\![B]\!])) \vdash p}\multimap L$$

At this point we can apply the "induction hypothesis" of the translation, asserting that the open premises follow since $\vdash \Sigma, A$ and $\vdash \Sigma, B$.

For $A \mathbin{⅋} B$, matters are a bit more complicated.

$$\dfrac{\vdash \Sigma, A, B}{\vdash \Sigma, A \mathbin{⅋} B}\,⅋$$

Since there is no $⅋$ connective on the intuitionistic side, we have to translate uses of the $⅋$ rule into application of the $\otimes L$ rule. This makes sense, since $⅋$ was justified as the formal dual of $\otimes$. This means we have to distribute the negations a bit differently.

$$[\![A \mathbin{⅋} B]\!] = {\sim}({\sim}[\![A]\!] \otimes {\sim}[\![B]\!])$$

Then we get (in somewhat abbreviated form)

$$\dfrac{\dfrac{\dfrac{\vdots}{{\sim}[\![\Sigma]\!], {\sim}[\![A]\!], {\sim}[\![B]\!]}}{{\sim}[\![\Sigma]\!], {\sim}[\![A]\!] \otimes {\sim}[\![B]\!]}\otimes L}{{\sim}[\![\Sigma]\!], {\sim}{\sim}({\sim}[\![A]\!] \otimes {\sim}[\![B]\!])}\multimap L, \multimap R$$

where the open subproof follows again inductively, from the translation of the premise of the classical $\parr$ rule.

We can continue to reason along these lines. For connectives where there is an intuitionistic counterpart, we just double-negate the subformulas. For those where there is not, we negate once, use the intuitionistic dual, and then negate once again. This leads us to the following table.

$$
\begin{array}{lcl}
[\![P]\!] & = & P \\
[\![P^\perp]\!] & = & \sim P \\[4pt]
[\![A \otimes B]\!] & = & \sim\sim[\![A]\!] \otimes \sim\sim[\![B]\!] \\
[\![A \parr B]\!] & = & \sim(\sim[\![A]\!] \otimes \sim[\![B]\!]) \\[4pt]
[\![\mathbf{1}]\!] & = & \mathbf{1} \\
[\![\perp]\!] & = & \sim\mathbf{1} \\[4pt]
[\![A \oplus B]\!] & = & \sim\sim[\![A]\!] \oplus \sim\sim[\![B]\!] \\
[\![A \,\&\, B]\!] & = & \sim\sim[\![A]\!] \,\&\, \sim\sim[\![B]\!] \\[4pt]
[\![\mathbf{0}]\!] & = & \mathbf{0} \\
[\![\top]\!] & = & \top \\[4pt]
[\![!A]\!] & = & !\sim\sim[\![A]\!] \\
[\![?A]\!] & = & \sim!\sim[\![A]\!]
\end{array}
$$

There are more economical translations where some double negations are omitted, but the one shown above seems most systematic.

6 Correctness of the Translation

From our little derivation, it is easy to see the following:

Theorem 1 (From CLL to ILL) *If* $\vdash \Sigma$ *then* $\sim[\![\Sigma]\!]_p \vdash p$.

Proof: By induction on the structure of the given derivation. A few lemmas are needed for the exponentials (see [CCP03]), to bridge the gap between the monadic and dyadic presentations of the logic. □

The converse requires an entirely different technique. First we observe that intuitionistic linear logic makes some finer distinctions (especially in the treatment of linear implication). If these distinctions are ignored, we can prove the result classically. In this translation, we think of $A \multimap B = A^\perp \parr B$, on the classical side.

Lemma 2 *If* $\Gamma \,;\, \Delta \rightarrow A$ *then* $\vdash (!\Gamma)^\perp, \Delta^\perp, A$

Proof: By induction on the given derivation, using some lemmas regarding classical provability. □

The second lemma we need is that, classically, the translation is essentially the identity, if we use $\perp$.

Lemma 3 *For any proposition* A*,* $[\![A]\!]_\perp \stackrel{CLL}{\equiv} A$*.*

Proof: A simple induction on the structure of A, mostly exploiting that $\sim_\perp\sim_\perp A \stackrel{CLL}{\equiv} A$. □

Theorem 4 (From ILL to CLL) *If* $\sim[\![\Sigma]\!]_p \vdash p$ *then* $\vdash \Sigma$*.*

Proof: If $\Sigma = A_1, \ldots, A_n$, we have

$$\sim_p[\![A_1]\!]_p, \ldots, \sim_p[\![A_n]\!]_p \vdash p$$

Since classical logic proves *more* (Lemma 2), we get

$$\vdash (\sim_p[\![A_1]\!]_p)^\perp, \ldots, (\sim_p[\![A_n]\!]_p)^\perp, p$$

This proof is parametric in p, so we can substitute $p = \perp$ throughout the proof and obtain

$$\vdash (\sim_\perp[\![A_1]\!]_\perp)^\perp, \ldots, (\sim_\perp[\![A_n]\!]_\perp)^\perp, \perp$$

Now we recall that $(\sim_\perp A)^\perp = (A^\perp \parr \perp)^\perp = (A \otimes \mathbf{1})$. Since $A \otimes \mathbf{1} \stackrel{CLL}{\equiv} A$ we can use cut multiple times and arrive at

$$\vdash [\![A_1]\!]_\perp, \ldots, [\![A_n]\!]_\perp, \perp$$

Then we recall that $\perp^\perp = \mathbf{1}$ so we can cut this with $\vdash \mathbf{1}$ to get

$$\vdash [\![A_1]\!]_\perp, \ldots, [\![A_n]\!]_\perp$$

Finally recall that $[\![A]\!]_\perp \stackrel{CLL}{\equiv} A$. Using cut A number of times we get

$$\vdash A_1, \ldots, A_n$$

which is what we needed to show. □

7 Mix and Other Variations

In his original paper [Gir87], Girard also discussed a variant of linear logic with the rules of mix. They are

$$\frac{}{\vdash \cdot}\ \mathsf{mix}_0 \qquad \frac{\vdash \Sigma \quad \vdash \Sigma'}{\vdash \Sigma, \Sigma'}\ \mathsf{mix}_2$$

It turns out that the logic with these roles (and good proof-theoretic properties), can also be characterized with axioms postulating that $\bot \overset{CLL}{\equiv} \mathbf{1}$. Why is that? If $\bot$ and $\mathbf{1}$ are equivalent, this means we have $\vdash \mathbf{1}, \mathbf{1}$ and $\vdash \bot, \bot$ since $\mathbf{1}^\bot = \bot$ and $\bot^\bot = \mathbf{1}$.

Then we can derive mix_0 as

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\ \mathbf{1} \quad \vdash \bot, \bot}{\vdash \bot}\ \mathsf{cut} \quad \dfrac{}{\vdash \mathbf{1}}\ \mathbf{1}}{\vdash \cdot}\ \mathsf{cut}$$

and mix_2 as

$$\dfrac{\dfrac{\vdash \Sigma}{\vdash \Sigma, \bot}\ \bot \quad \dfrac{\dfrac{\vdash \Sigma'}{\vdash \Sigma', \bot}\ \bot \quad \vdash \mathbf{1}, \mathbf{1}}{\vdash \Sigma', \mathbf{1}}\ \mathsf{cut}}{\vdash \Sigma, \Sigma'}\ \mathsf{cut}$$

Now we can proceed as in the previous section, and exploit the parametricity of the translation by using

$$p = \mathbf{1}$$

In the crucial step, we use

$$(\sim_{\mathbf{1}} A)^\bot = (A^\bot \mathbin{⅋} \mathbf{1})^\bot = A \otimes \bot \overset{MIX}{\equiv} A \otimes \mathbf{1}$$

In this way we come to the conclusion that using the mix rules in the classical setting is just like trying consume all linear resources (proving $\mathbf{1}$), rather than trying to derive a contradiction (proving $\bot$). Since the process calculus admits such as interpretation, it seems reasonable that in encodings of concurrent computation the mix rule is difficult to deny. In the intuitionistic case, we can derive a counterpart as follows.

A process that does not offer any external services, has the form

$$\Gamma \,;\, \Delta \vdash P :: z : \mathbf{1}$$

Two such processes can be combined as follows:

$$\dfrac{\Gamma \,;\, \Delta \vdash P :: z : \mathbf{1} \quad \dfrac{\Gamma \,;\, \Delta' \vdash Q :: w : \mathbf{1}}{\Gamma \,;\, \Delta', z{:}\mathbf{1} \vdash z().Q :: w : \mathbf{1}}\,\mathbf{1}L}{\Gamma \,;\, \Delta, \Delta' \vdash (\nu z)(P \mid z().Q) :: w : \mathbf{1}}\,\mathsf{cut}$$

This, however, does not quite have the desired effect, because Q cannot reduce until P has completed its computation. It is, in effect, a sequential composition. This is why, in most recent incarnations of the proof term assignment, we have separated the input prefix from its scope. In the earliest paper [CP10], the $\mathbf{1}L$ rule was entirely silent, but that created some small discord between the proof theory and the process reductions, as attested by the relative complexity of the bisimulation theorems in that paper. With the newer process assignment we obtain:

$$\dfrac{\Gamma \,;\, \Delta \vdash P :: z : \mathbf{1} \quad \dfrac{\Gamma \,;\, \Delta' \vdash Q :: w : \mathbf{1}}{\Gamma \,;\, \Delta', z{:}\mathbf{1} \vdash z().\mathbf{0} \mid Q :: w : \mathbf{1}}\,\mathbf{1}L}{\Gamma \,;\, \Delta, \Delta' \vdash (\nu z)(P \mid z().\mathbf{0} \mid Q) :: w : \mathbf{1}}\,\mathsf{cut}$$

This now permits P and Q to proceed in parallel.

We can replace p by other constants and obtain other interpretations. At this point, one of them is still open, in the sense that we have not found a good independent proof-theoretically satisfying characterization (see [CCP03]).

Exercises

Exercise 1 (Classical Harmony) Give the missing cut reductions and identity expansions for classical linear logic.

Exercise 2 (Dyadic Classical Linear Logic) Give the rules for a one-sided, two-zone sequent calculus based on the same ideas as separating persistent resources from linear ones. Show that derivable sequents are the same as the ones for the one-sided, one-zone sequent calculus presented in this lecture.

Exercise 3 (Mix) Prove that in the presence of the mix rules, $\vdash \mathbf{1}, \mathbf{1}$ and $\vdash \bot, \bot$ are derivable.

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