Chapter 6

Labeled Deduction

Starting from a system of natural deduction for the definition of intuitionistic logic, we have made a remarkable journey, including the sequent calculus, focusing, and the inverse method. Many, if not all of the idea are shared between many reasonable and useful logics: intuitionistic logic, classical logic, linear logic, modal logic, temporal logic, and probably many more. In this chapter we see another surprisingly robust idea: labeled deduction. There are many views of labeled deduction. One of the most general is that we relativize our notion of truth. While intuistionistic logic is based on a single unary judgment, namely $A\,\mathit{true}$, labeled deduction is based on binary judgments of the form $A\,\mathit{true}[p]$, where p is a *label* or *world*. We may read $A\,\mathit{true}[p]$ as "*A is true at world p.*"

The uses of a relativized notions of truth are many; we concentrate here only on a single one. The motivation comes from developing a sequent calculus for intuitionistic logic in which all rules are invertible. Alternatively, it can be seen as a means of interpreting intuitionistic logic in classical logic (we have already seen the opposite). Wallen's book [Wal90] is the seminal work in this area with respect to automated deduction and is still fresh after more than a decade. A newer reference is Waaler's article in Handbook of Automated Reasoning [Waa01]. Often cited is also Fitting's book [Fit83], but it seems to be difficult to obtain.

6.1 Multiple Conclusions

One of the problems with focusing is that disjunction on the right-hand side is opaque: if we have a conclusion $A \vee B$ may have to try to prove A or B and then backtrack to prove the other without sharing of information between the attempts. Moreover, while focusing on a left synchronous formula, we completely ignore the shape of the succedent. An idea to remedy this situation is to replace $A \vee B$ by A, B on the right-hand side, postponing the choice between A and B. It is difficult to give a satisfactory judgmental reading of multiple propositions on the right, but let us suspend this issue and simply read A, B on the right as a *postponed choice* between A and B.

Our basic judgment form is now

$$\Gamma \stackrel{m}{\Longrightarrow} \Delta$$

to be read as "*Under assumptions* Γ *prove one of* Δ*,*" although it will not be the case that there is always one element in Δ that we can actually prove. Initial sequents, conjunction, and disjunction are as in the judgment for classical logic, $\Gamma \# \Delta$, in which Γ are assumptions about truth and Δ assumptions about falsehood.

$$\frac{}{\Gamma, P \stackrel{m}{\Longrightarrow} P, \Delta}\,\text{init}$$

$$\frac{\Gamma, A, B \stackrel{m}{\Longrightarrow} \Delta}{\Gamma, A \wedge B \stackrel{m}{\Longrightarrow} \Delta}\,{\wedge}\text{L} \qquad \frac{\Gamma \stackrel{m}{\Longrightarrow} A, \Delta \qquad \Gamma \stackrel{m}{\Longrightarrow} B, \Delta}{\Gamma \stackrel{m}{\Longrightarrow} A \wedge B, \Delta}\,{\wedge}\text{R}$$

$$\frac{\Gamma, A \stackrel{m}{\Longrightarrow} \Delta \qquad \Gamma, B \stackrel{m}{\Longrightarrow} \Delta}{\Gamma, A \vee B \stackrel{m}{\Longrightarrow} \Delta}\,{\vee}\text{L} \qquad \frac{\Gamma \stackrel{m}{\Longrightarrow} A, B, \Delta}{\Gamma \stackrel{m}{\Longrightarrow} A \vee B, \Delta}\,{\vee}R$$

Since we have already observed that conjunction and disjunction are really the same for intuitionistic and classical logic, perhaps the rules above do not come as a suprise. But how to we salvage the intuitionistic nature of the logic? Consider the problem of $(A \supset B) \vee A$, which is classically true for all A and B, but not intuitionistically. The classical proof is

$$\cfrac{\cfrac{\cfrac{}{A \# B, A}\,\text{init}}{\cdot \# (A \supset B), A}\,{\supset}F}{\cdot \# (A \supset B) \vee A}\,{\vee}F$$

If we try to interpret this proof intuitionistically, replacing $\#$ by $\stackrel{m}{\Longrightarrow}$, we see that the right rule for implication looks very suspicious: the scope of the assumption A should be B (since we say: $A \supset B$), and yet it appears to include the other disjunct, A. In this way we avoid ever producing evidence for one of the propositions on the right: we exploit one to prove the other.

To avoid this counterexample, we have to change the implication right rule to be the following:

$$\frac{\Gamma, A \supset B \stackrel{m}{\Longrightarrow} A, \Delta \qquad \Gamma, B \stackrel{m}{\Longrightarrow} \Delta}{\Gamma, A \supset B \stackrel{m}{\Longrightarrow} \Delta}\,{\supset}\text{L} \qquad \frac{\Gamma, A \stackrel{m}{\Longrightarrow} B}{\Gamma \stackrel{m}{\Longrightarrow} A \supset B, \Delta}\,{\supset}\text{R}$$

The crucial point is that before we can use $\supset$R we have to commit a choice to preserve the scope of the new assumption A. This sequent calculus admits weakening and contraction on both sides and a cut elimination theorem. It is also sound and complete, although a theorem to that effect must be formulated carefully.

Before that, we can add the logical constants for truth and falsehood.

$$\frac{\Gamma \overset{m}{\Longrightarrow} \Delta}{\Gamma, \top \overset{m}{\Longrightarrow} \Delta}\,\top L \qquad \frac{}{\Gamma \overset{m}{\Longrightarrow} \top, \Delta}\,\top R$$

$$\frac{}{\Gamma, \bot \overset{m}{\Longrightarrow} \Delta}\,\bot L \qquad \frac{\Gamma \overset{m}{\Longrightarrow} \Delta}{\Gamma \overset{m}{\Longrightarrow} \bot, \Delta}\,\bot R$$

Negation can be derived from implication and falsehood.

$$\frac{\Gamma, \neg A \overset{m}{\Longrightarrow} A, \Delta}{\Gamma, \neg A \overset{m}{\Longrightarrow} \Delta}\,\neg L \qquad \frac{\Gamma, A \overset{m}{\Longrightarrow} \cdot}{\Gamma \overset{m}{\Longrightarrow} \neg A, \Delta}\,\neg R$$

Note that $\neg$R makes a commitment, erasing Δ, as for implication.

The first, natural idea at soundness would state that if $\Gamma \overset{m}{\Longrightarrow} \Delta$, then there is a proposition C in Δ such that $\Gamma \Longrightarrow C$. This, unfortunately, is false, as can be seen from $A \vee B \overset{m}{\Longrightarrow} B, A$ is is provable and, yet, neither B or A by itself follows from $A \vee B$. We write $\bigvee(A_1, \ldots, A_n)$ for $A_1 \vee \cdots \vee A_n$ which is interpreted as $\bot$ if $n = 0$.

Theorem 6.1 (Soundness of Multiple-Conclusion Sequent Calculus) *If* $\Gamma \overset{m}{\Longrightarrow} \Delta$ *then* $\Gamma \Longrightarrow \bigvee \Delta$.

Proof: By induction on the given derivation. Most cases are immediate. We show only the implication cases.

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma, A \overset{m}{\Longrightarrow} B\end{array}}{\Gamma \overset{m}{\Longrightarrow} A \supset B, \Delta}\,\supset R$$

$\Gamma, A \Longrightarrow B$	By i.h.
$\Gamma \Longrightarrow A \supset B$	By rule $\supset$R
$\Gamma \Longrightarrow (A \supset B) \vee \bigvee \Delta$	By repeated $\vee R$

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma, A \supset B \overset{m}{\Longrightarrow} A, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma, B \overset{m}{\Longrightarrow} \Delta\end{array}}{\Gamma, A \supset B \overset{m}{\Longrightarrow} \Delta}\,\supset L$$

$\Gamma, A \supset B \Longrightarrow A \vee C$ for $C = \bigvee \Delta$	By i.h.
$\Gamma, B \Longrightarrow C$	By i.h.
$\Gamma \Longrightarrow B \supset C$	By rule $\supset$R
$\Gamma, A \supset B, B \supset C, A \vee C \Longrightarrow C$	Direct proof
$\Gamma, A \supset B, B \supset C \Longrightarrow C$	By admissibility of cut
$\Gamma, A \supset B \Longrightarrow C$	By admissibility of cut

□

Theorem 6.2 (Completness of Multiple-Conclusion Sequent Calculus) *If* $\Gamma \Longrightarrow A$ *then* $\Gamma \stackrel{m}{\Longrightarrow} A$

Proof: By induction on the given derivation. Most cases are immediate. In the case of $\vee R$ we need to apply weakening after the induction hypothesis. □

6.2 Propositional Labeled Deduction

The next problem is to avoid or at least postpone the choice associated with the $\supset$R rule. However, it is clear we cannot simply leave Δ around, since this would yield classical logic, as the example in the previous section demonstrates. Instead we label assumptions and conclusion in such a way that the new assumption A will be prohibited from being used in the proof of any proposition in the conclusion except for its natural scope, B. In other words, we enforce scoping by labeling. We need *label parameters* $a, b, \ldots$ and *labels*, where a label is simply a sequence of label parameters.

$$\text{Labels} \quad p, q \quad ::= \quad a_1\, a_2 \ldots a_n$$

We use ϵ to denote the empty sequence of labels. An assumption $A\ \mathit{true}[p]$ is supposed to be available to prove any conclusion $B\ \mathit{true}[pq]$, that is, the scope of any label includes any extension of that label. We abbreviate $A\ \mathit{true}[p]$ as $A[p]$. Initial sequents then have the form

$$\frac{}{\Gamma, A[p] \Longrightarrow A[pq], \Delta}\,\text{init}$$

In the implication right rule we create a new scope, by introducing a new label parameter.

$$\frac{\Gamma, A[pa] \Longrightarrow B[pa], \Delta}{\Gamma \Longrightarrow (A \supset B)[p], \Delta}\,{\supset}\mathrm{R}^a$$

Important is that the parameter a must be new. Therefore, for no conlusion $C[q]$ in Δ could q be an extension of pa. Effectively, the scope of $A[pa]$ excludes Δ.

Revisiting an earlier example (and anticipating that $\vee$ propagates its labels to both subformulas), we see that it is not provable because ϵ is not an extension of a.

$$\dfrac{\dfrac{\overline{A[a] \Longrightarrow B[a], A[\epsilon]}\;?}{\cdot \Longrightarrow (A \supset B)[\epsilon], A[\epsilon]}\supset\mathrm{R}^a}{\cdot \Longrightarrow (A \supset B) \vee A[\epsilon]}\vee R$$

The implication left rule incorporates the fact that an assumption $(A \supset B)[p]$ is available in any extension of p. When we apply $\supset$L we have to choose the world in which we can show $A[pq]$. It is in this world that we can assume $B[pq]$.

$$\dfrac{\Gamma, (A \supset B)[p] \Longrightarrow A[pq] \qquad \Gamma, B[pq] \Longrightarrow \Delta}{\Gamma, (A \supset B)[p] \Longrightarrow \Delta}\supset\mathrm{L}$$

As an example, consider the beginning of the proof of transitivity.

$$\dfrac{\dfrac{\dfrac{A \supset B[a], B \supset C[ab], A[abc] \Longrightarrow C[abc]}{A \supset B[a], B \supset C[ab] \Longrightarrow A \supset C[ab]}\supset\mathrm{R}^c}{A \supset B[a] \Longrightarrow (B \supset C) \supset A \supset C[a]}\supset\mathrm{R}^b}{\Longrightarrow (A \supset B) \supset (B \supset C) \supset (A \supset C)[\epsilon]}\supset\mathrm{R}^a$$

At this point we have to apply implication left to either $A \supset B[a]$ or $B \supset C[ab]$. The difficulty is to guess at which extended label to apply it. If we apply the $\supset$L rule to $A \supset B[a]$ we can we see we must be able to prove $A[aq]$ for some q. But we have available only $A[abc]$, so q must be an extension of bc.

$$\dfrac{\dfrac{}{A \supset B[a], B \supset C[ab], A[abc] \Longrightarrow A[abc]}\mathrm{init} \qquad B \supset C[ab], A[abc], B[abc] \Longrightarrow C[abc]}{A \supset B[a], B \supset C[ab], A[abc] \Longrightarrow C[abc]}\supset\mathrm{L}$$

We continue in the right premise with another implication left rule, this time choosing $q = c$ so we can prove $B[abq]$.

$$\dfrac{\dfrac{}{B \supset C[ab], A[abc], B[abc] \Longrightarrow B[abc]}\mathrm{init} \qquad \dfrac{}{A[abc], B[abc], C[abc] \Longrightarrow C[abc]}\mathrm{init}}{B \supset C[ab], A[abc], B[abc] \Longrightarrow C[abc]}\supset\mathrm{L}$$

In the rules for remaining propositional connectives, the labels do not change because no new scope is introduced.

$$\dfrac{\Gamma, A[p], B[p] \Longrightarrow \Delta}{\Gamma, (A \wedge B)[p] \Longrightarrow \Delta}\wedge\mathrm{L} \qquad \dfrac{\Gamma \Longrightarrow A[p], \Delta \qquad \Gamma \Longrightarrow B[p], \Delta}{\Gamma \Longrightarrow (A \wedge B)[p], \Delta}\wedge\mathrm{R}$$

$$\dfrac{\Gamma, A[p] \Longrightarrow \Delta \qquad \Gamma, B[p] \Longrightarrow \Delta}{\Gamma, (A \vee B)[p] \Longrightarrow \Delta}\vee\mathrm{L} \qquad \dfrac{\Gamma \Longrightarrow A[p], B[p], \Delta}{\Gamma \Longrightarrow (A \vee B)[p], \Delta}\vee R$$

Truth and falsehood are also straightforward.

$$\frac{\Gamma \Longrightarrow \Delta}{\Gamma, \top[p] \Longrightarrow \Delta}\,\top\mathrm{L} \qquad \frac{}{\Gamma \Longrightarrow \top[p], \Delta}\,\top\mathrm{R}$$

$$\frac{}{\Gamma, \bot[p] \Longrightarrow \Delta}\,\bot\mathrm{L} \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \bot[p], \Delta}\,\bot\mathrm{R}$$

A way to think about the $\bot$L rule is to consider that $\bot[p]$ entails the empty right-hand side from which we can generate Δ by weakening. So it makes sense even if all the worlds in Δ are out of the scope defined by p. We can determine the laws for negation from considerations for implication and falsehood.

$$\frac{\Gamma, (\neg A)[p] \Longrightarrow A[pq], \Delta}{\Gamma, (\neg A)[p] \Longrightarrow \Delta}\,\neg\mathrm{L} \qquad \frac{\Gamma, A[pa] \Longrightarrow \Delta}{\Gamma \Longrightarrow (\neg A)[p], \Delta}\,\neg\mathrm{R}^a$$

The $\neg$R rule is subject to the proviso that a does not appear in the conclusion.

Showing the soundness and completeness of labeled deduction is not a trivial enterprise; we show here only completeness. A critical notion is that of a *monotone sequent*. We write $p \preceq q$ if there exists a p' such that $p\,p' = q$ and say *p is a prefix of q*. We say a sequent $A_1[p_1], \ldots, A_n[p_n] \Longrightarrow C_1[q_1], \ldots, C_m[q_m]$ is *monotone at q* if $q_j = q$ for all $1 \leq j \leq m$ and every p_i is a prefix of q, that is, $p_i \preceq q$ for all $1 \leq i \leq m$.

Theorem 6.3 (Completeness of Labeled Deduction) *If $\Gamma \stackrel{m}{\Longrightarrow} \Delta$ is derivable then for any monotone labeling $\Gamma' \Longrightarrow \Delta'$ of $\Gamma \stackrel{m}{\Longrightarrow} \Delta$, we have that $\Gamma' \Longrightarrow \Delta'$ is derivable.*

Proof: By induction on the structure of the given derivation. We show a few cases.

Case:

$$\mathcal{D} = \frac{}{\Gamma, P \stackrel{m}{\Longrightarrow} P, \Delta}\,\mathrm{init}$$

$\Gamma', P[p] \Longrightarrow P[q], \Delta'$ monotone at q	Assumption
$p \preceq q$	By defn. of monotonicity
$\Gamma', P[p] \Longrightarrow P[q], \Delta'$	By rule init

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \stackrel{m}{\Longrightarrow} B\end{array}}{\Gamma \stackrel{m}{\Longrightarrow} A \supset B, \Delta}\,{\supset}\mathrm{R}$$

$\Gamma' \Longrightarrow (A \supset B)[q], \Delta'$ monotone at q	Assumption
$\Gamma', A[qa] \Longrightarrow B[qa]$ monotone at qa for a new a	By defn. of monotonicity
$\Gamma', A[qa] \Longrightarrow B[qa]$ derivable	By i.h.
$\Gamma' \Longrightarrow (A \supset B)[q]$ derivable	By rule $\supset\mathrm{R}^a$
$\Gamma' \Longrightarrow (A \supset B)[q], \Delta'$ derivable	By weakening

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma, A \supset B \stackrel{m}{\Longrightarrow} A, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma, B \stackrel{m}{\Longrightarrow} \Delta\end{array}}{\Gamma, A \supset B \stackrel{m}{\Longrightarrow} \Delta} \supset\text{L}$$

$\Gamma', A \supset B[p] \Longrightarrow \Delta'$ monotone at q	Assumption
$\Gamma', A \supset B[p] \Longrightarrow A[q], \Delta'$ monotone at q	By defn. of monotonicity
$\Gamma', A \supset B[p] \Longrightarrow A[q], \Delta'$ derivable	By i.h.
$\Gamma', B[q] \Longrightarrow \Delta'$ monotone at q	By defn. of monotonicity
$\Gamma', B[q] \Longrightarrow \Delta'$ derivable	By i.h.
$\Gamma', (A \supset B)[p] \Longrightarrow \Delta'$	By rule $\supset$L and $p \preceq q$

$\square$

The soundness proof is considerably more difficult. Standard techniques are via so-called Kripke models or by direct translation from matrix proofs to the sequent calculus. On of the problems is that the (unlabeled) proof will generally have to proceed with a different order of the inferences than the labeled proof. The interested reader is refered to Wallen [Wal90], Waaler [Waa01], and Schmitt et al. [KS00, SLKN01].

6.3 First-Order Labeled Deduction

In first-order intuitionistic logic, it is not just the implication that introduces a new scope, but also universal quantification. This means we have to change both the multiple-conclusion sequent calculus and the labeled deduction system. The changes in the multiple-conclusion calculus is quite straightforward; the change to the labeled calculus are more extensive. We show here only the rules, but not any proofs. The reader is refered to the literature cited at the beginning of this chapter for details.

$$\dfrac{\Gamma, \forall x.\ A(x), A(t) \stackrel{m}{\Longrightarrow} \Delta}{\Gamma, \forall x.\ A(x) \stackrel{m}{\Longrightarrow} \Delta} \forall\text{L} \qquad \dfrac{\Gamma \stackrel{m}{\Longrightarrow} A(b)}{\Gamma \stackrel{m}{\Longrightarrow} \forall x.\ A(x), \Delta} \forall\text{R}^b$$

$$\dfrac{\Gamma, A(b) \stackrel{m}{\Longrightarrow} \Delta}{\Gamma, \exists x.\ A(x) \stackrel{m}{\Longrightarrow} \Delta} \exists\text{L}^b \qquad \dfrac{\Gamma \stackrel{m}{\Longrightarrow} A(t), \exists x.\ A(x), \Delta}{\Gamma \stackrel{m}{\Longrightarrow} \exists x.\ A(x), \Delta} \exists\text{R}$$

The side condition on $\forall\text{R}^b$ and $\exists\text{L}^b$ is the usual: b must not occur in the conclusion. Note that Δ is erased in the premise of $\forall$R, and that an extra copy of $\exists x.\ A(x)$ is kept in the $\exists$R rule.

The fact that universal quantification creates a new scope means that in the labeled deductive systems, terms must now also be labeled. We have a new judgment t *term*$[p]$ which means *t is a well-formed term at p*. We may abbreviate this as $t[p]$. We introduce a new set of assumptions in order to track the labels at which they have been introduced.

$$\text{Labeled Parameter Contexts} \quad \Sigma \quad ::= \quad \cdot \mid \Sigma, a\ \mathit{term}[p]$$

We have two principal judgments.

$$\begin{array}{l} \Sigma; \Gamma \Longrightarrow \Delta \\ \Sigma \vdash t\ \mathit{term}[p] \end{array}$$

The first just adds an explicit parameter context to a sequent, the second test whether terms are well-formed. The latter is defined by the following rules:

$$\frac{a\ \mathit{term}[p] \text{ in } \Sigma}{\Sigma \vdash a\ \mathit{term}[pq]}\,\text{parm} \qquad \frac{\Sigma \vdash t_i\ \mathit{term}[p] \quad \text{for all } 1 \leq i \leq n}{\Sigma \vdash f(t_1, \ldots t_n)\ \mathit{term}[p]}\,\text{func}$$

As propositional assumptions, term assumptions remain valid in future worlds (allowing pq in the parameter rule). In the rules for $\Sigma; \Gamma \Longrightarrow \Delta$, Σ is carried through from conclusion to premises in all rules except those containing quantifiers. The new rules for quantifiers are:

$$\frac{\Sigma \vdash t[pq] \qquad \Sigma; \Gamma, \forall x.\ A(x)[p], A(t)[pq] \Longrightarrow \Delta}{\Sigma; \Gamma, \forall x.\ A(x)[p] \Longrightarrow \Delta}\,\forall\text{L}$$

$$\frac{\Sigma, b[pa]; \Gamma \Longrightarrow A(b)[pa], \Delta}{\Sigma; \Gamma \Longrightarrow \forall x.\ A(x)[p], \Delta}\,\forall\text{R}^{b,a}$$

$$\frac{\Sigma, b[p]; \Gamma, A(b)[p] \Longrightarrow \Delta}{\Sigma; \Gamma, \exists x.\ A(x)[p] \Longrightarrow \Delta}\,\exists\text{L}^{b}$$

$$\frac{\Sigma \vdash t[p] \qquad \Sigma; \Gamma \Longrightarrow A(t)[p], \exists x.\ A(x)[p], \Delta}{\Sigma; \Gamma \Longrightarrow \exists x.\ A(x)[p], \Delta}\,\exists\text{R}$$

6.4 Matrix Methods

The system of labeled deduction, if propositional or first-order, still has non-invertible rules. Specifically, implication and universal quantification on the left are synchronous, as well as existential quantification on right. These propositions may have to wait for a label or term parameter to be introduced before they can be decomposed.

In order to postpone these choices we can introduce free variables, standing both for labels and terms, and employ unification (again, both for labels and terms) for possibly initial sequents. These kinds of algorithms are usually described as so-called *matrix methods*, *connections methods*, or *mating methods*, originally developed for classical logic.

This is a large subject, and we forego a special treatment here. A good introduction, with further pointers to the literature, can be found in Waaler's article [Waa01] in the *Handbook of Automated Reasoning*. Highly recommended is also Wallen's book [Wal90], although it does not fully address some of the more difficult aspects of the implementation such as label unification.

Bibliography

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

[Byr99] John Byrnes. *Proof Search and Normal Forms in Natural Deduction*. PhD thesis, Department of Philosophy, Carnegie Mellon University, May 1999.

[Cur30] H.B. Curry. Grundlagen der kombinatorischen Logik. *American Journal of Mathematics*, 52:509–536, 789–834, 1930.

[DV99] Anatoli Degtyarev and Andrei Voronkov. Equality reasoning in sequent-based calculi. In Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning*. Elsevier Science Publishers, 1999. In preparation.

[Fit83] Melvin Fitting. *Proof Methods for Modal and Intuitionistic Logics*. D.Reidel Publishing Co., Dordrecht, 1983.

[Gen35] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. Translated under the title *Investigations into Logical Deductions* in [Sza69].

[Her30] Jacques Herbrand. Recherches sur la théorie de la démonstration. *Travaux de la Société des Sciences et de Lettres de Varsovic*, 33, 1930.

[Her95] Hugo Herbelin. *Séquents qu'on calcule*. PhD thesis, Universite Paris 7, January 1995.

[Hil22] David Hilbert. Neubegründung der Mathematik (erste Mitteilung). In *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, pages 157–177, 1922. Reprinted in [Hil35].

[Hil35] David Hilbert. *Gesammelte Abhandlungen*, volume 3. Springer-Verlag, Berlin, 1935.

[How69] W. A. Howard. The formulae-as-types notion of construction. Unpublished manuscript, 1969. Reprinted in To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, 1980.

[How98] Jacob M. Howe. *Proof Search Issues in Some Non-Classical Logics*. PhD thesis, University of St. Andrews, Scotland, 1998.

[Hua94] Xiarong Huang. *Human Oriented Proof Presentation: A Reconstructive Approach*. PhD thesis, Universität des Saarlandes, Saarbrücken, Germany, 1994. Published by infix, St. Augustin, Germany, Dissertationen zur Künstlichen Intelligenz, Volume 112, 1996.

[Hue76] Gérard Huet. *Résolution d'équations dans des langages d'ordre* $1, 2, \ldots, \omega$. PhD thesis, Université Paris VII, September 1976.

[Kle52] Stephen Cole Kleene. *Introduction to Metamathematics*. North-Holland, 1952.

[Kni89] Kevin Knight. Unification: A multi-disciplinary survey. *ACM Computing Surveys*, 2(1):93–124, March 1989.

[KS00] Christoph Kreitz and Stephan Schmitt. A uniform procedure for converting matrix proofs into sequent-style systems. *Information and Computation*, 162(1–2):226–254, 2000.

[LS86] Joachim Lambek and Philip J. Scott. *Introduction to Higher Order Categorical Logic*. Cambridge University Press, Cambridge, England, 1986.

[Mas64] S. Maslov. The inverse method of establishing deducibility in the classical predicate calculus. *Soviet Mathematical Doklady*, 5:1420–1424, 1964.

[Min94] G. Mints. Resolution strategies for the intuitionistic logic. In *Constraint Programming*, pages 289–311. NATO ASI Series F, Springer-Verlag, 1994.

[ML85a] Per Martin-Löf. On the meanings of the logical constants and the justifications of the logical laws. Technical Report 2, Scuola di Specializzazione in Logica Matematica, Dipartimento di Matematica, Università di Siena, 1985.

[ML85b] Per Martin-Löf. Truth of a proposition, evidence of a judgement, validity of a proof. Notes to a talk given at the workshop *Theory of Meaning*, Centro Fiorentino di Storia e Filosofia della Scienza, June 1985.

[ML94] Per Martin-Löf. Analytic and synthetic judgements in type theory. In Paolo Parrini, editor, *Kant and Contemporary Epistemology*, pages 87–99. Kluwer Academic Publishers, 1994.

[MM76] Alberto Martelli and Ugo Montanari. Unification in linear time and space: A structured presentation. Internal Report B76-16, Istituto di Elaborazione delle Informazione, Consiglio Nazionale delle Ricerche, Pisa, Italy, July 1976.

[MM82] Alberto Martelli and Ugo Montanari. An efficient unification algorithm. *ACM Transactions on Programming Languages and Systems*, 4(2):258–282, April 1982.

[Par92] Michel Parigot. $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction. In A. Voronkov, editor, *Proceedings of the International Conference on Logic Programming and Automated Reasoning*, pages 190–201, St. Petersburg, Russia, July 1992. Springer-Verlag LNCS 624.

[Pfe95] Frank Pfenning. Structural cut elimination. In D. Kozen, editor, *Proceedings of the Tenth Annual Symposium on Logic in Computer Science*, pages 156–166, San Diego, California, June 1995. IEEE Computer Society Press.

[Pra65] Dag Prawitz. *Natural Deduction*. Almquist & Wiksell, Stockholm, 1965.

[PW78] M. S. Paterson and M. N. Wegman. Linear unification. *Journal of Computer and System Sciences*, 16(2):158–167, April 1978.

[Rob65] J. A. Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–41, January 1965.

[Rob71] J. A. Robinson. Computational logic: The unification computation. *Machine Intelligence*, 6:63–72, 1971.

[SLKN01] Stephan Schmitt, Lori Lorigo, Christoph Kreitz, and Alexey Nogin. Jprover: Integrating connection-based theorem proving into interactive proof assistants. In R.Goré, A.Leitsch, and T.Nipkow, editors, *Proceedings of the International Joint Conference on Automated Reasoning (IJCAR'01)*, pages 421–426, Siena, Italy, June 2001. Springer Verlag LNAI 2083.

[Sza69] M. E. Szabo, editor. *The Collected Papers of Gerhard Gentzen*. North-Holland Publishing Co., Amsterdam, 1969.

[Tam96] T. Tammet. A resolution theorem prover for intuitionistic logic. In M. McRobbie and J. Slaney, editors, *Proceedings of the 13th International Conference on Automated Deduction (CADE-13)*, pages 2–16, New Brunswick, New Jersey, 1996. Springer-Verlag LNCS 1104.

[Tam97] T. Tammet. Resolution, inverse method and the sequent calculus. In A. Leitsch G. Gottlog and D. Mundici, editors, *Proceedings of the 5th Kurt Gödel Colloquium on Computational Logic and Proof Theory (KGC'97)*, pages 65–83, Vienna, Austria, 1997. Springer-Verlag LNCS 1289.

[Vor92] Andrei Voronkov. Theorem proving in non-standard logics based on the inverse method. In D. Kapur, editor, *Proceedings of the 11th International Conference on Automated Deduction*, pages 648–662, Saratoga Springs, New York, 1992. Springer-Verlag LNCS 607.

[Vor96] Andrei Voronkov. Proof-search in intuitionistic logic with equality, or back to simultaneous rigid e-unification. In M.A. McRobbie and J.K. Slaney, editors, *Proceedings of the 13th International Conference on Automated Deduction*, pages 32–46, New Brunswick, New Jersey, July/August 1996. Springer-Verlag LNAI 1104.

[Waa01] Arild Waaler. Connections in nonclassical logics. In Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning*, volume 2, chapter 22, pages 1487–1578. Elsevier Science and MIT Press, 2001.

[Wal90] Lincoln A. Wallen. *Automated Deduction in Non-Classical Logics*. MIT Press, 1990.