

Transformations

Vector Spaces
Affine and Euclidean Spaces
Frames
Homogeneous Coordinates
Transformation Matrices
[Angel, Ch. 4]

January 23, 2003
Frank Pfenning
Carnegie Mellon University
<http://www.cs.cmu.edu/~fp/courses/graphics/>

Geometric Objects and Operations

- Primitive types: scalars, vectors, points
- Primitive operations: dot product, cross product
- Representations: coordinate systems, frames
- Implementations: matrices, homogeneous coord.
- Transformations: rotation, scaling, translation
- Composition of transformations
- OpenGL transformation matrices

01/23/2003

15-462 Graphics I

2

Scalars

- Scalars α, β, γ from *scalar field*
- Operations $\alpha + \beta, \alpha \cdot \beta, 0, 1, -\alpha, ()^{-1}$
- “Expected” laws apply
- Examples: rationals or reals with addition and multiplication

01/23/2003

15-462 Graphics I

3

Vectors

- Vectors u, v, w from *vector space*
- Includes scalar field
- Vector addition $u + v$
- Zero vector $\mathbf{0}$
- Scalar multiplication αv



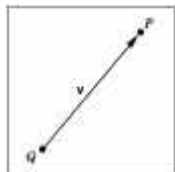
01/23/2003

15-462 Graphics I

4

Points

- Points P, Q, R from *affine space*
- Includes vector space
- Point-point subtraction $v = P - Q$
- Define also $P = v + Q$



01/23/2003

15-462 Graphics I

5

Euclidean Space

- Assume vector space over real numbers
- Dot product: $\alpha = u \cdot v$
- $\mathbf{0} \cdot \mathbf{0} = 0$
- u, v are *orthogonal* if $u \cdot v = 0$
- $|v|^2 = v \cdot v$ defines $|v|$, the *length* of v
- Generally work in an affine Euclidean space

01/23/2003

15-462 Graphics I

6

Geometric Interpretations

- Lines and line segments
- Convexity
- Dot product and projections
- Cross product and normal vectors
- Planes

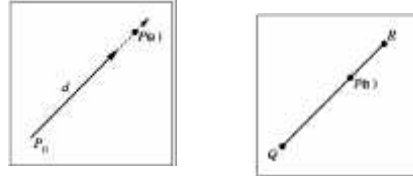
01/23/2003

15-462 Graphics I

7

Lines and Line Segments

- Parametric form of line: $P(\alpha) = P_0 + \alpha d$



- Line segment between Q and R:

$$P(\alpha) = (1-\alpha)Q + \alpha R \text{ for } 0 \leq \alpha \leq 1$$

01/23/2003

15-462 Graphics I

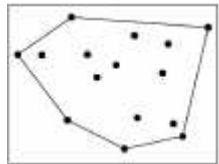
8

Convex Hull

- Convex hull defined by

$$P = \alpha_1 P_1 + \dots + \alpha_n P_n$$

for $\alpha_1 + \dots + \alpha_n = 1$
and $0 \leq \alpha_i \leq 1, i = 1, \dots, n$



01/23/2003

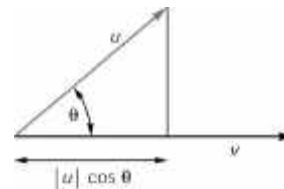
15-462 Graphics I

9

Projection

- Dot product projects one vector onto other

$$u \cdot v = |u| |v| \cos(\theta)$$



01/23/2003

15-462 Graphics I

10

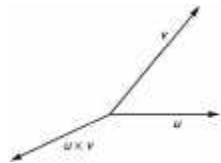
Normal Vector

- Cross product defines normal vector

$$u \times v = n$$

$$|u \times v| = |u| |v| |\sin(\theta)|$$

- Right-hand rule



01/23/2003

15-462 Graphics I

11

Plane

- Plane defined by point P_0 and vectors u and v
- u and v cannot be parallel
- Parametric form: $T(\alpha, \beta) = P_0 + \alpha u + \beta v$
- Let $n = u \times v$ be the normal
- Then $n \cdot (P - P_0) = 0$ iff P lies in plane

01/23/2003

15-462 Graphics I

12

Outline

- Vector Spaces
- Affine and Euclidean Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices
- OpenGL Transformation Matrices

01/23/2003

15-462 Graphics I

13

Coordinate Systems

- Let v_1, v_2, v_3 be three linearly independent vectors in a 3-dimensional vector space

- Can write any vector w as

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

for scalars $\alpha_1, \alpha_2, \alpha_3$

- In matrix notation:

$$\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

01/23/2003

15-462 Graphics I

14

Frames

- Frame = coordinate system + origin P_0
- Any point $P = P_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
- Useful in with homogenous coordinates

01/23/2003

15-462 Graphics I

15

Changes of Coordinate System

- Bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$
- Express basis vectors u_i in terms of v_j

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$

$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$

$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

- Represent in matrix form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{for } \mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

01/23/2003

15-462 Graphics I

16

Map to Representations

- $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, $\mathbf{a}^T = [\alpha_1 \alpha_2 \alpha_3]$

- $w = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$, $\mathbf{b}^T = [\beta_1 \beta_2 \beta_3]$

$$\mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = w = \mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{b}^T \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

- So $\mathbf{a} = \mathbf{M}^T \mathbf{b}$ and $\mathbf{b} = (\mathbf{M}^T)^{-1} \mathbf{a}$
- Suffices for rotation and scaling, not translation

01/23/2003

15-462 Graphics I

17

Outline

- Vector Spaces
- Affine and Euclidean Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices
- OpenGL Transformation Matrices

01/23/2003

15-462 Graphics I

18

Linear Transformations

- 3×3 matrices represent linear transformations
- $\mathbf{a} = \mathbf{M} \mathbf{b}$
- Can represent rotation, scaling, and reflection
- Cannot represent translation
- \mathbf{a} and \mathbf{b} represent vectors, not points

$$\mathbf{w} = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

01/23/2003

15-462 Graphics I

19

Homogeneous Coordinates

- In affine space, $P = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + P_0$
- Define $0 \cdot P = 0$, $1 \cdot P = P$
- Then

$$P = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 1] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

- Point $\mathbf{p} = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 1]^T$
- Vector $\mathbf{w} = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3$
- Homogeneous coords: $\mathbf{a} = [\delta_1 \quad \delta_2 \quad \delta_3 \quad 0]^T$

01/23/2003

15-462 Graphics I

20

Translation of Frame

- Express frame (u_1, u_2, u_3, P_0) in (v_1, v_2, v_3, Q_0)

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$

$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$

$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

$$Q_0 = \gamma_{41}v_1 + \gamma_{42}v_2 + \gamma_{43}v_3 + P_0$$

- Then

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} \text{ for } \mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

01/23/2003

15-462 Graphics I

21

Homogeneous Coordinates Summary

- Points $[\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 1]^T$
- Vectors $[\delta_1 \quad \delta_2 \quad \delta_3 \quad 0]^T$
- Change of frame

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

01/23/2003

15-462 Graphics I

22

Outline

- Vector Spaces
- Affine and Euclidean Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices
- OpenGL Transformation Matrices

01/23/2003

15-462 Graphics I

23

Affine Transformations

- Translation
- Rotation
- Scaling
- Any composition of the above
- Express in homogeneous coordinates
- Need 4×4 matrices
- Later: projective transformations
- Also expressible as 4×4 matrices!

01/23/2003

15-462 Graphics I

24

Translation

- $\mathbf{p}' = \mathbf{p} + \mathbf{d}$ where $\mathbf{d} = [\alpha_x \ \alpha_y \ \alpha_z \ 0]^T$
- $\mathbf{p} = [x \ y \ z \ 1]^T$
- $\mathbf{p}' = [x' \ y' \ z' \ 1]^T$
- $x' = x + \alpha_x, y' = y + \alpha_y, z' = z + \alpha_z$
- Express in matrix form $\mathbf{p}' = \mathbf{T} \mathbf{p}$ and solve for \mathbf{T}

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

01/23/2003

15-462 Graphics I

25

Scaling

- $x' = \beta_x x$
- $y' = \beta_y y$
- $z' = \beta_z z$
- Express as $\mathbf{p}' = \mathbf{S} \mathbf{p}$ and solve for \mathbf{S}

$$\mathbf{S} = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

01/23/2003

15-462 Graphics I

26

Rotation in 2 Dimensions

- Rotation by θ about the origin
- $x' = x \cos \theta - y \sin \theta$
- $y' = x \sin \theta + y \cos \theta$
- Express in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Note determinant is 1

01/23/2003

15-462 Graphics I

27

Rotation in 3 Dimensions

- Decompose into rotations about x, y, z axes

$$\mathbf{R}_z = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

01/23/2003

15-462 Graphics I

28

Compose by Matrix Multiplication

- $\mathbf{R} = \mathbf{R}_z \mathbf{R}_y \mathbf{R}_x$
- Applied from right to left
- $\mathbf{R} \mathbf{p} = (\mathbf{R}_z \mathbf{R}_y \mathbf{R}_x) \mathbf{p} = \mathbf{R}_z (\mathbf{R}_y (\mathbf{R}_x \mathbf{p}))$
- "Postmultiplication" in OpenGL

01/23/2003

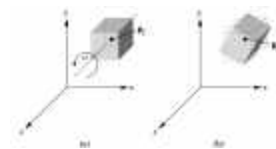
15-462 Graphics I

29

Rotation About a Fixed Point

- First, translate to the origin
- Second, rotate about the origin
- Third, translate back
- To rotate by θ about z around \mathbf{p}_f

$$\mathbf{M} = \mathbf{T}(\mathbf{p}_f) \mathbf{R}_z(\theta) \mathbf{T}(-\mathbf{p}_f) = \dots$$



01/23/2003

15-462 Graphics I

30

Deriving Transformation Matrices

- Other examples: see [Angel, Ch. 4.8]
- See also Assignment 2 when it is out
- Hint: manipulate matrices, but remember geometric intuition

01/23/2003

15-462 Graphics I

31

Outline

- Vector Spaces
- Affine and Euclidean Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices
- OpenGL Transformation Matrices

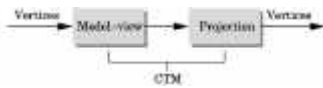
01/23/2003

15-462 Graphics I

32

Current Transformation Matrix

- Model-view matrix (usually affine)
- Projection matrix (usually not affine)



- Manipulated separately

```
glMatrixMode(GL_MODELVIEW);  
glMatrixMode(GL_PROJECTION);
```

01/23/2003

15-462 Graphics I

33

Manipulating the Current Matrix

- Load or postmultiply

```
glLoadIdentity();  
glLoadMatrixf(*m);  
glMultMatrixf(*m);
```

- Library functions to compute matrices

```
glTranslatef(dx, dy, dz);  
glRotatef(angle, vx, vy, vz);  
glScalef(sx, sy, sz);
```

- Recall: last transformation is applied first!

01/23/2003

15-462 Graphics I

34

Summary

- Vector Spaces
- Affine and Euclidean Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices
- OpenGL Transformation Matrices

01/23/2003

15-462 Graphics I

35

OpenGL Tutors by Nate Robins

- Run under Windows
- Available at <http://www.xmission.com/~nate/tutors.html>
- Example: [Transformation tutor](#)

01/23/2003

15-462 Graphics I

36