Reduced Power Automata

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Abstract. We describe a class of transitive semiautomata whose power automata are reduced: any two reachable sets of states have distinct behavior. These automata appear naturally in the study of one-dimensional cellular automata.

1 Motivation

The acceptance languages of transitive semiautomata enjoy a number of special properties. Notably, they are factorial, extensible and transitive (FET): $uv \in L \Longrightarrow u, v \in L$, $u \in L \Longrightarrow \exists\, a, b \in \Sigma\,(aub \in L)$, and $u, v \in L \Longrightarrow \exists\, x\,(uxv \in L)$ where Σ denotes the underlying alphabet). For languages of this type there is an alternative notion of minimal deterministic automaton, first introduced by Fischer [5] and discovered independently by Beauquier [3] in the form of the 0-minimal ideal in the syntactic semigroup of L. A *Fischer* automaton is a deterministic transitive semiautomaton. For each factorial, extensible and transitive language there is a unique Fischer automaton that minimizes the number of states. Thus, for any FET language L we can measure the state complexity in two ways: as the size $\mu(L)$ of the standard minimal DFA for L, or as the size $\mu_F(L)$ of the minimal Fischer automaton for L. The minimal Fischer automaton naturally embeds into the ordinary minimal deterministic automaton, see [9]. Thus, except in the trivial case where $L = \Sigma^*$, we have $\mu_F(L) \leq \mu(L) - 1$ and one can compute the minimal Fischer automaton in linear time given the standard minimal DFA. Note that $\mu_F(L)$ is arguably a better measure for the complexity of L than $\mu(L) - 1$ since the minimal Fischer automaton can be quite small even when the minimal DFA is large, see [11].

However, there is a fundamental obstruction to computing $\mu(L)$ or $\mu_F(L)$ for a FET language L: these languages are often given as a nondeterministic transitive semiautomaton $\mathcal{A}$. If the semiautomaton has size n, the only a priori bounds available are $\mu_F(L) \leq \mu(L) - 1$ and $\mu(L) \leq 2^n$ since the accessible part $\text{pow}(\mathcal{A})$ of the Rabin-Scott power automaton of $\mathcal{A}$ has size at most 2^n. We write $\pi(\mathcal{A})$ for the size of this automaton. One can construct the minimal automaton in polynomial time from $\text{pow}(\mathcal{A})$, but as shown in [11] it is PSPACE-hard to determine whether $\pi(\mathcal{A})$ is less than a given bound. Hence there is no feasible computational shortcut that would allow one to determine the size of the power automaton without actually constructing the machine. Moreover, [9, 11] show that exponential blow-up where $\pi(\mathcal{A})$ is equal to or close to the upper bound 2^n occurs quite frequently.

We are particularly interested in the languages that arise in the study of one-dimensional cellular automata, see [14, 7, 2, 12, 6]. For our purposes here, a cellular automaton can be represented as a local map $\rho : \Sigma^w \to \Sigma$ that extends naturally to a global map on bi-infinite words on the alphabet Σ, usually referred to as configurations in this context. The cover of a configuration is the set of all its finite factors. It is easy to see that $\text{cov}(\rho)$, the union of all covers of $\rho(X)$ where X ranges over all configurations, is a regular language. Indeed, the natural semiautomaton for this language is a de Bruijn automaton $B(\rho)$ whose state set is Σ^{w-1} and whose transitions are of the form $(ax, \rho(axb), xb)$ where $a, b \in \Sigma$ and $x \in \Sigma^{w-2}$, see [8, 10, 4]. The same holds for the cover languages $\text{cov}(\rho^t)$ associated with the iterates of the global map. The de Bruijn automaton here has size $k^{t(w-1)}$ where k denotes the size of the alphabet Σ. Thus, even for binary alphabets the only obvious upper bound is

$$\mu(\text{cov}(\rho^t)) \leq 2^{2^{t(w-1)}}.$$

In the mid eighties, Wolfram performed extensive calculations in an effort to understand the behavior of the sequences $\mu(\text{cov}(\rho^t))$, see [13–15]. As shown in [9] the doubly exponential upper bound can be reached for any width w at time $t = 1$, though none of the iterates ρ^t, $t > 1$, display full blow-up.

One can characterize the cellular automata ρ for which full blow-up occurs as follows. Define a *1-permutation* automaton to be any transitive semiautomaton that is obtained by changing the label of a single transition in a permutation automaton. In a permutation automaton each symbol in Σ

induces a permutation of the state set; equivalently, the automaton is deterministic, codeterministic and complete. When the selected transition is a loop we refer to the automaton as a *loop-1-permutation* automaton. For de Bruijn automaton $\mathsf{B}(\rho)$ full blow-up occurs only for 1-permutation automata. For loop-1-permutation automata we have a particularly simple situation, see [9].

Theorem 1. *Let ρ be a binary cellular automaton such that $\mathsf{B}(\rho)$ is a loop-1-permutation automaton of size $n = 2^{w-1}$. Then $\mu(\mathcal{A}) = \pi(\mathcal{A}) = 2^n$ and $\mu_F(\mathcal{A}) = \mu(\mathcal{A}) - 1$.*

In this paper we extend this result in two ways. First, we show that full blow-up occurs in all degenerate 1-permutation automata, see below for definitions. Second, we demonstrate that in general for any 1-permutation automaton $\mathcal{A}$ we have $\mu(\mathcal{A}) = \pi(\mathcal{A})$, regardless of the actual size of the power automaton. Hence, given the power automaton $\mathrm{pow}(\mathcal{A})$ we can compute $\mu_F(\mathcal{A})$ in linear time. In the next section we briefly introduce some terminology, but in general we refer the reader to the references for more background information. Section 3 contains the main result, and in the last section we comment on open problems.

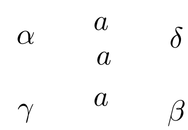
2 One-Permutation Automata

Let $\mathcal{A} = \langle Q, \Sigma, \cdot \rangle$ be any automaton, p a state in $\mathcal{A}$. We will often abuse notation and write p rather than $\{p\}$. We denote $[\![p]\!]_{\mathcal{A}}$ the *behavior* of state p in $\mathcal{A}$, i.e., the set of words accepted if p is chosen as initial state. Likewise $[\![P]\!]_{\mathcal{A}}$ for $P \subseteq Q$ denotes the set of words accepted if P is chosen as set of initial states. Recall that an automaton *synchronizes (on state p)* if there exists a word w such that $Q \cdot w = p$. $\mathcal{A}$ *synchronizes completely* if it synchronizes on all its states. A non-empty set $P \subseteq Q$ is *rich* in $\mathcal{A}$ if $\bigcap_{p \in P} [\![p]\!]_{\mathcal{A}} - [\![Q - P]\!]_{\mathcal{A}} \neq \emptyset$. The automaton is rich if all its states are rich. The reversal $\mathcal{A}^{\mathrm{op}}$ of an automaton is obtained by replacing all transitions $p \xrightarrow{s} q$ by $q \xrightarrow{s} p$. Similarly, x^{op} denotes the reversal of a word x. Note that $\mathcal{A}^{\mathrm{op}}$ is a 1-permutation automaton whenever $\mathcal{A}$ is. Since $x \in [\![q]\!]_{\mathcal{A}}$ if, and only if, $q \in Q \cdot x^{\mathrm{op}}$ in $\mathcal{A}^{\mathrm{op}}$ we have the following proposition.

Proposition 1. *State p is rich in $\mathcal{A}$ if, and only if, $\mathcal{A}^{op}$ synchronizes on p. More generally, $P \subseteq Q$ is rich in $\mathcal{A}$ if, and only if, P is reachable in $\mathcal{A}^{op}$.*

Hence, if $\mathcal{A}^{\mathrm{op}}$ synchronizes completely the full power automaton of $\mathcal{A}$ is reduced, and its accessible part $\mathrm{pow}(\mathcal{A})$ is the minimal automaton, so $\pi(\mathcal{A}) = \mu(\mathcal{A})$.

To simplify notation, let $\mathcal{A}_0$ be a transitive permutation automaton over the alphabet $\{a, b\}$ and fix a transition $\tau = (\alpha, b, \beta)$ in $\mathcal{A}_0$. We denote $\mathcal{A}$ the 1-permutation automaton obtained by flipping the label of that transition to a. Thus, locally the 1-permutation automaton $\mathcal{A}$ has the following structure.



Proposition 2. *Let $\mathcal{A}$ be a 1-permutation automaton. Then $\mathcal{A}$ synchronizes on α, and β is rich in $\mathcal{A}$.*

Proof. Suppose $P \subseteq Q$ has cardinality larger than 1. Define the standard length-lex order on words to be the product order where words a first compared by length, and then each group of words of the same length is ordered lexicographically. Let x be length-lex minimal such that $\alpha \in P \cdot x$. Then $|\, P \cdot xb \,| = |\, P \,| - 1$ and $\mathcal{A}$ synchronizes on α by induction. Since $\mathcal{A}^{\mathrm{op}}$ is also a 1-permutation automaton it follows by the same argument and proposition 1 that β is rich in $\mathcal{A}$. □

Proposition 1 allows us to demonstrate blow-up of machine $\mathcal{A}$ by arguing about the richness of sets of states in $\mathcal{A}^{\mathrm{op}}$. To this is end it is convenient to think of computations as moving pebbles according to some input sequence. For example, to show that p is rich we can place a red pebble on p and a black pebble on each state in $Q - p$. We then have to remove all black pebbles (by moving them to α and then firing a b transition), without losing the red one. A loss could occur because the red pebble is located at α and the next symbol is b, or because the red pebble moves to β, but a black pebble arrives there at the same time. Of course, both red and black pebbles will split when located at α and the next symbol is an a, so there may be several red pebbles after a while.

To demonstrate this approach, consider an automaton $\mathcal{A}$ defined on a circulant graph $C(n; 1, d)$ where $1 < d < n$. Figure 1 shows a 1-permutation automaton based on $C(6; 1, 2)$.

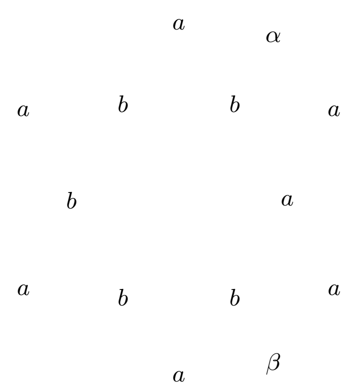


Fig. 1. A 1-permutation automaton on the circulant graph $C(6; 1, 2)$.

In this case, $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are isomorphic. To see that an arbitrary non-empty set $P \subseteq Q$ is rich place red pebbles on all the elements of P, and black pebbles on all the states in $Q - P$. We can now fire a sequence of input symbols a until a black pebble is moved to α, and then fire b. Note that no red pebbles are lost, even if the second incarnation of a pebble being placed onto β is eliminated by a black pebble arriving there at the same time. Hence, we can remove all black pebbles. It follows that $\pi(\mathcal{A}) = 2^n$ and, as we will see shortly, $\mu(\mathcal{A}) = 2^n$. However, the size of the minimal Fischer automaton is $m(2^{n/m} - 1)$ where $m = \gcd(n, d - 1)$: the states in $P \subseteq Q$ reachable from a single state p all have distances a multiple of $d - 1$, and all such P are indeed reachable.

As a first step towards our main result, let us first dispense with the case when two of the states α, β, δ or γ coincide. Let us call a 1-permutation automaton *degenerate* if either $\alpha = \beta$ or $\delta = \gamma$. Thus loop-1-permutation automata in particular are degenerate, and degenerate 1-permutation automata locally have the following structure.

Theorem 2. *Let $\mathcal{A}$ be a degenerate 1-permutation automaton of size n. Then $\mu(\mathcal{A}) = \pi(\mathcal{A}) = 2^n$.*

Proof. By theorem 1 we only have to deal with the case $\delta = \gamma$. In theorem 3 we will show that $\mu(\mathcal{A}) = \pi(\mathcal{A})$, so it suffices to show that in $\mathcal{A}^{\mathrm{op}}$ every subset of Q is rich. But $\mathcal{A}^{\mathrm{op}}$ is again degenerate, so to simplify notation we will show that every subset of Q is rich in $\mathcal{A} = \mathcal{A}^{\mathrm{op\,op}}$. To this end we show that for every R disjoint from P we have $\bigcap_{r \in R} [\![r]\!]_{\mathcal{A}} - [\![P]\!]_{\mathcal{A}} \neq \emptyset$.

Let A be the a-labeled cycle in $\mathcal{A}_0$ that contains the consecutive edges $\alpha \xrightarrow{a} \delta = \gamma \xrightarrow{a} \beta$. Note that in $\mathcal{A}$ there is a chord $\alpha \xrightarrow{a} \beta$ in this cycle. Define a partial order on the power set of Q by $P \prec P'$ if $|\,P\,| < |\,P'\,|$, or if $|\,P\,| = |\,P'\,|$ and the length-lex minimal word x such that $\alpha \in P \cdot x$ precedes the corresponding word for P', again in length-lex order.

Suppose P is $\prec$-minimal such that for some $R \cap P = \emptyset$ we have $\bigcap_{r \in R} [\![r]\!]_{\mathcal{A}} \subseteq [\![P]\!]_{\mathcal{A}}$. Place red pebbles on R and black pebbles on P accordingly. By the minimality of P we must have $\alpha \notin P$. Let x be the minimal witness such that $\alpha \in P \cdot x$. Then $x = bu$ and there is a red pebble on α, otherwise there would be a violation of the minimality of P.

Let m be the least common multiple of the lengths of all a-labeled cycles in $\mathcal{A}_0$. Move the pebbles according to a^m. Since P is minimal, no black pebble can be on A, so that all black pebbles return to their original positions. The red pebble originally at α now has a clone at δ. Hence we can now use b to move the black pebbles closer to α without destroying a red pebble (and all its descendants). Since $|\,P\,| = |\,P \cdot a^m b\,|$ this contradicts our minimality assumption. ☐

Note that no claims are made about the size of the minimal Fischer automaton in the last theorem. It is not true in general that $\mu_F(\mathcal{A}) = 2^n - 1$, though in the special case of de Bruijn automata this relation obtains: in this case $\gamma = \delta = 0^k$ or 1^k, so there is a loop at $\gamma = \delta$ labeled b. From this it follows easily that $\mathcal{A}$ synchronizes on β, thus Q is reachable from α in $\mathcal{A}$.

3 Reduced Power Automata

We will now show that the power automata arising from 1-permutation automata are always reduced. $\mathcal{A}^-$ denotes the reversible automaton obtained by removing the transition τ from $\mathcal{A}_0$.

Lemma 1. *Let $\mathcal{A}$ be a 1-permutation automaton where both (α, a, δ) and (γ, a, β) lie in the same strongly connected component of $\mathcal{A}^-$. Then $\mathcal{A}$ is rich.*

Proof. First consider a cycle C of the form $\beta \longrightarrow \alpha \xrightarrow{a} \delta \longrightarrow \gamma \xrightarrow{a} \beta$. We claim that every point q on C is rich. To see this let $P \subseteq Q$ where $q \notin P$, and place red and black pebbles on Q correspondingly. We may safely assume that $q \neq \beta$ by proposition 2.

If there is no black pebble on α fire symbol s to advance the pebble from q one step towards β. If there is a black pebble on α pick $i > 0$ such that either $q \cdot b^i = q$ or $q \cdot b^i = \alpha$ depending on whether the red pebble is currently located on a b-cycle or on the open path labeled b ending at α. In both cases, we either remove at least one black pebble or we bring the red pebble closer to β on C. Our claim follows by induction.

But then every state is rich: we can move a red pebble from anywhere to the cycle C. $\square$

Accessibility plays no role in the last argument, so we have the following corollary.

Corollary 1. *Let $\mathcal{A}$ be as in the last lemma. Then the full power automaton of $\mathcal{A}$ is reduced. In particular, $\mu(\mathcal{A}) = \pi(\mathcal{A})$.*

From now on assume that (α, a, δ) and (γ, a, β) lie in two separate transitive subautomata $\mathcal{A}_\delta$ and $\mathcal{A}_\beta$ determined by the strongly connected components of δ and β in $\mathcal{A}^-$. A typical example of this situation is shown in figure 2. The subautomaton $\mathcal{A}_\delta$ here has states $\{\alpha, \delta\}$ and all other states lie in $\mathcal{A}_\beta$.

In this automaton the behavior of δ is strictly contained in the behavior of $\{\beta, \beta'\}$, and the full power automaton fails to be reduced. However, we will see that $\{\beta, \beta'\}$ is not reachable, and the power automaton is still

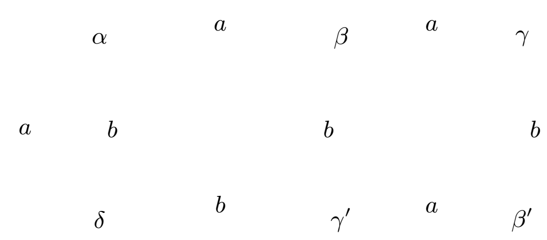


Fig. 2. A 1-permutation automaton with δ covered by $\{\beta, \beta'\}$.

reduced. To see why, consider $P_1 \neq P_2 \subseteq Q$ but $[\![P_1]\!]_{\mathcal{A}} = [\![P_2]\!]_{\mathcal{A}}$. Without loss of generality assume $p \in P_1 - P_2$. Let x be length-lex minimal such that $p \cdot x = \alpha$. Since β is rich we must have $\gamma \in P_2 \cdot x$: otherwise $\beta \in p \cdot xa$ but $\beta \notin P_2 \cdot xa$. Moreover, $\delta \in p \cdot xa$ and we must have $\beta \in P_2 \cdot xau$ for any u such that $\delta \cdot u = \delta$. This is captured in the next definition. $P \subseteq Q$ *covers* δ if $\delta \notin P$ but $[\![\delta]\!]_{\mathcal{A}} \subseteq [\![P]\!]_{\mathcal{A}}$. A cover P is *minimal* if it is minimal with respect to cardinality.

Thus, covers form the essential obstruction to the full power automaton being reduced. We will now describe the structure of minimal covers in great detail so as to show that these sets cannot appear in the accessible part of the power automaton.

Proposition 3. *Let P be a minimal cover of δ and x in the behavior of δ. Then $|\,P \cdot x\,| = |\,P\,|$.*

Proof. By minimality, the cardinality of $P \cdot x$ cannot decrease for any $x \in [\![\delta]\!]_{\mathcal{A}}$. Assume for the sake of a contradiction that $|\,P \cdot x\,| > |\,P\,|$. Let $u \in [\![\delta]\!]_{\mathcal{A}}$ be the minimal prefix of x such that $\alpha \in P \cdot u$. Note that $\alpha \neq \delta \cdot u$ since no merge can occur on α. But then $ub \in [\![\delta]\!]_{\mathcal{A}}$ whereas $|\,P \cdot ub\,| < |\,P\,|$, contradiction. □

Proposition 4. *Let P be a minimal cover of δ and $\delta = \delta \cdot x$. Then $P \cdot x$ is a minimal cover and x acts like a permutation on P.*

Proof. We have $[\![\delta]\!]_{\mathcal{A}} = x^{-1}[\![\delta]\!]_{\mathcal{A}} \subseteq x^{-1}[\![P]\!]_{\mathcal{A}}$ and therefore $[\![\delta]\!]_{\mathcal{A}} \subseteq [\![P \cdot x]\!]_{\mathcal{A}}$. The first claim follows from the last proposition.

Since $\delta = \delta \cdot x^i$ for all $i \geq 0$ it follows that $P \cdot x^i$ is a minimal cover for all i. But then $P \cdot x^i = P \cdot x^{i+j}$ for some $i \geq 0$, $j > 0$. Hence the orbits are labeled x^j and our claim follows. □

Lemma 2. *Let P be a minimal cover of δ. Then*

$$P = \{ \beta \cdot x \mid \delta \cdot x = \delta \text{ in } \mathcal{A}^- \}$$

Proof. Denote $P_0 = \{ \beta \cdot x \mid \delta \cdot x = \delta \text{ in } \mathcal{A}^- \}$ and pick $x \neq \varepsilon$ such that $\delta \cdot x = \delta$ in $\mathcal{A}^-$. Clearly $x = ua$ and, by proposition 4, x permutes P. But $\delta \cdot u = \alpha$, so $\gamma \in P \cdot u$ and $\beta \in P \cdot x$. It follows that $P_0 \subseteq P$ and it suffices to show that P_0 is a cover. So suppose u is in the behavior of δ. Then for some suitable suffix v we have $\delta \in \delta \cdot (uv)^i$ for all i, and P_0 contains a cycle

$$\beta = \beta_0 \xrightarrow{uv} \beta_1 \xrightarrow{uv} \beta_2 \xrightarrow{uv} \dots \xrightarrow{uv} \beta_{r-1} \xrightarrow{uv} \beta_r = \beta.$$

But then u is in the behavior of P_0. □

Since both subautomata $\mathcal{A}_\delta$ and $\mathcal{A}_\beta$ are reversible we have a natural partial action of the free group over $\{a, b\}$ on their state sets, see [1] for details. To avoid confusion with our π function we write $\mathbb{F}(\mathcal{A}_\delta, \delta)$ for the fundamental group of $\mathcal{A}_\delta$, and $\mathbb{F}(\mathcal{A}_\beta, P)$ for the words in the free group over $\{a, b\}$ that permute P.

Lemma 3. *Let P be a minimal cover of δ. Then $\mathbb{F}(\mathcal{A}_\delta, \delta) = \mathbb{F}(\mathcal{A}_\beta, P)$.*

Proof. We first show $\mathbb{F}(\mathcal{A}_\delta, \delta) \subseteq \mathbb{F}(\mathcal{A}_\beta, P)$. We have $\{a, b\}^* \cap \mathbb{F}(\mathcal{A}_\delta, \delta) \subseteq \mathbb{F}(\mathcal{A}_\beta, P)$ from proposition 4. For the sake of simplicity, we consider only the case where $x \in \mathbb{F}(\mathcal{A}_\delta, \delta)$ contains only one symbol $\overline{a}$, the general case is entirely analogous. So, assume $x = u\overline{a}w$ where $u, w \in \{a, b\}^*$. Since $\mathcal{A}_\delta$ is strongly connected we can choose a word $v \in \{a, b\}^*$ such that $\delta \cdot uw = \delta \cdot u\overline{a}$. But then $u(wa)^i wv$ permutes P for all $i \geq 0$. For any $p \in P$ set $q_p = p \cdot u$, so that $q_p \cdot wu = f(p)$ for some permutation f. wa permutes the set $\{ q_p \mid p \in P \}$. Hence, $q_p \cdot \overline{a} = q_p \cdot (wa)^i w$ for some $i \geq 0$ and it follows that $u\overline{a}v$ permutes P.

For the opposite direction let $x \in \mathbb{F}(\mathcal{A}_\beta, P)$ and set $q = \delta \cdot x$. Since $\beta \in P \cdot x$ we must have $q = \delta \cdot b^i$ for some $i \geq 0$, so that $x\overline{b}^i \in \mathbb{F}(\mathcal{A}_\delta, \delta) \subseteq \mathbb{F}(\mathcal{A}_\beta, P)$. But then $x\overline{b}^i$ permutes P, whence $i = 0$ and we have $x \in \mathbb{F}(\mathcal{A}_\delta, \delta)$. □

Theorem 3. *Let $\mathcal{A}$ be a 1-permutation automaton. Then the accessible part of the power automaton is reduced. Hence $\mu(\mathcal{A}) = \pi(\mathcal{A})$.*

Proof. By lemma 1 we may safely assume that (α, a, β) and (γ, a, δ) do not lie in the same strongly connected component of $\mathcal{A}^-$.

First consider the case when δ has no cover, i.e., when δ is rich. Since there is a path $p \xrightarrow{x} \delta$ from any state p other than β that avoids β it follows that $\mathcal{A}$ is rich and we are done as in corollary 1.

So suppose $P_1 \neq P_2 \subseteq Q$ but $[\![P_1]\!]_{\mathcal{A}} = [\![P_2]\!]_{\mathcal{A}}$. Without loss of generality assume $p \in P_1 - P_2$. As mentioned previously, we must have $p \cdot xa = \delta$ and $P_2 \cdot xa$ covers δ. But covers are not reachable. To see this, assume otherwise, say $P = Q \cdot x$ for some cover P. Since $\delta \notin P$ and all states in ${\mathcal{A}_\delta}^{\text{op}}$ are complete, x must be of the form $x = vbu$ where $\delta' \cdot bu = \delta$. But then $u \in \mathbb{F}(\mathcal{A}_\delta)$, so $P = Q \cdot vb$ by lemma 3, contradicting the fact that β is in P by lemma 2.

Hence P_2 cannot be reachable either, and we are done. □

4 Open Problems

As pointed out in the introduction, any cellular automaton whose associated FET language has maximum complexity must be a 1-permutation automaton. Some of these automata are accounted for by theorem 2. However, computational experiments suggest that exactly half of the 1-permutation automata associated with binary cellular automata of width w demonstrate full blow-up:

Conjecture 1. There are $2^{w-1}2^{2^{w-2}}$ 1-permutation automata de Bruijn automata $\mathcal{A}$ such that

$$\mu(\mathcal{A}) = \mu_F(\mathcal{A}) + 1 = 2^{2^{w-1}}.$$

It seems that the size of the power automaton of a 1-permutation automaton $\mathcal{A}$ is usually invariant under reversal, the automaton in figure 2 being a case in point. There are exceptions, but we are unable to characterize them at this point.

Conjecture 2. In general, for a 1-permutation automaton $\mathcal{A}$ we have $\mu(\mathcal{A}) = \mu(\mathcal{A}^{\text{op}})$.

Generalizing 1-permutation automata to k-permutation automata in the obvious way one can see that the corresponding cellular automata tend to have smaller μ and π values with increasing k. We do not presently understand the nature of this correlation.

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