Soundness of labelled deduction
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1 Introduction

This proof of the soundness of labelled deduction is based on the Kripke-model completeness proof from section 5.3 of Dirk van Dalen's *Logic and Structure*. It takes from van Dalen the idea of using prime contexts, but makes that notion more constructive by replacing the requirement of deductive closure with deductive closure under subformulas. Also, rather than finding a prime extension of a context that preserves a non-entailment so as to build a countermodel, we construct a *set* of prime extensions that fully characterizes entailment for the original context, and avoid model theory altogether.

2 Prime Contexts

Definition A context Γ is said to be **prime** if it meets the following conditions:

1. if A occurs as a subformula in Γ and $\Gamma \vdash A$ then $A \in \Gamma$
2. if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$
3. $\bot \notin \Gamma$

Lemma 2.1 *If Γ is prime then the following hold:*

1. *if $\Gamma \vdash A \vee B$ then $\Gamma \vdash A$ or $\Gamma \vdash B$ (disjunction property)*
2. *$\Gamma \nvdash \bot$ (consistency)*

Proof: The proofs of both facts make essential use of the subformula property. The derivation of $\Gamma \vdash A \vee B$ must end in either $\vee R$ or a left rule, and a potential derivation of $\Gamma \vdash \bot$ could only end in a left rule. Since the disjunction property holds immediately when the derivation ends in $\vee R$, the only interesting cases are the left rules. We consider only the disjunction property in the case of the $\vee L$ rule (the other cases—for the consistency property as well—are similar):

$$\frac{\Gamma, C \vee D, C \vdash A \vee B \quad \Gamma, C \vee D, D \vdash A \vee B}{\Gamma, C \vee D \vdash A \vee B}$$

Since $\Gamma, C \vee D$ is prime, either $C \in \Gamma$ or $D \in \Gamma$, so suppose (w.l.o.g.) it is C. Then $\Gamma, C \vee D$ is just a contraction of $\Gamma, C \vee D, C$, so in particular the latter is prime and we can invoke the i.h. to get that either $\Gamma, C \vee D, C \vdash A$ or $\Gamma, C \vee D, C \vdash B$. After applying contraction, we get the desired result. ■

Lemma 2.2 *Let Γ be a context. There exists a set S of prime contexts such that for all formulas A, $\Gamma \vdash A$ if and only if $\forall \Gamma' \in S.\ \Gamma' \vdash A$. We call S the* **set of prime extensions** *of Γ.*

Proof: We construct S by iterating the following procedure, starting with the set $S := \{\Gamma\}$:

- If S contains a context Γ' with a subformula A such that $\Gamma' \vdash A$ but $A \notin \Gamma'$, then replace Γ' by Γ', A in S.
- If S contains a context Γ' such that $A \vee B \in \Gamma'$ but neither $A \in \Gamma'$ nor $B \in \Gamma'$, then replace Γ' in S by two contexts, Γ', A and Γ', B.
- If S contains a context Γ' such that $\bot \in \Gamma'$, then remove Γ' from S.

Since Γ has only finitely many subformulas, this construction eventually terminates with S being a set (possibly empty) of prime contexts. Moreover, S has the property that for any A, $\Gamma \vdash A$ if and only $\forall \Gamma' \in S.\ \Gamma' \vdash A$, since that property is preserved by each step of the construction. ■

3 Soundness of labelled deduction

The labelled sequent calculus is defined like so:

$$\frac{}{\Gamma, P[p] \vdash^{\ell} P[pq], \Delta}\ init$$

$$\frac{\Gamma, A[p], B[p] \vdash^{\ell} \Delta}{\Gamma, A \wedge B[p] \vdash^{\ell} \Delta}\ \wedge L \qquad \frac{\Gamma \vdash^{\ell} A[p], \Delta \quad \Gamma \vdash^{\ell} B[p], \Delta}{\Gamma \vdash^{\ell} A \wedge B[p], \Delta}\ \wedge R$$

$$\frac{\Gamma, A[p] \vdash^{\ell} \Delta \quad \Gamma, B[p] \vdash^{\ell} \Delta}{\Gamma, A \vee B[p] \vdash^{\ell} \Delta}\ \vee L \qquad \frac{\Gamma \vdash^{\ell} A[p], B[p], \Delta}{\Gamma \vdash^{\ell} A \vee B[p], \Delta}\ \vee R$$

$$\frac{\Gamma, A \supset B[p] \vdash^{\ell} A[pq], \Delta \quad \Gamma, B[pq] \vdash^{\ell} \Delta}{\Gamma, A \supset B[p] \vdash^{\ell} \Delta}\ \supset L \qquad \frac{\Gamma, A[pa] \vdash^{\ell} B[pa], \Delta}{\Gamma \vdash^{\ell} A \supset B[p], \Delta}\ \supset R^{a}$$

$$\frac{}{\Gamma, \bot[p] \vdash^{\ell} \Delta}\ \bot L$$

Definition Let W be a set of worlds. A **realization** of W is a map $\mathfrak{R}$ that assigns to each world $p \in W$ a prime context $\mathfrak{R}(p)$, which moreover respects the accessibility relation on W in that $\mathfrak{R}(p) \subseteq \mathfrak{R}(pq)$.

Theorem 3.1 (Realization) *Suppose $\Gamma \vdash^{\ell} \Delta$, and assume $\mathfrak{R}$ is a realization of the worlds in Γ and Δ. Then if $\mathfrak{R}(p) \vdash A$ for each hypothesis $A[p] \in \Gamma$, there must exist a $B[q] \in \Delta$ such that $\mathfrak{R}(q) \vdash B$.*

Proof: By induction on the derivation of $\Gamma \vdash^{\ell} \Delta$.

- (*init*) By assumption $\mathfrak{R}(p) \vdash P$ and $\mathfrak{R}(p) \subseteq \mathfrak{R}(pq)$, which implies $\mathfrak{R}(pq) \vdash P$ by weakening.
- ($\wedge L$) Immediate, since $\mathfrak{R}(p) \vdash A \wedge B$ implies $\mathfrak{R}(p) \vdash A$ and $\mathfrak{R}(p) \vdash B$, and we invoke the i.h. on the unique premise.
- ($\wedge R$) We apply the i.h. to the two premises and obtain that either there exists a $B'[q'] \in \Delta$ such that $\mathfrak{R}(q') \vdash B'$ (in which case we are done) or else both $\mathfrak{R}(p) \vdash A$ and $\mathfrak{R}(p) \vdash B$, from which $\mathfrak{R}(p) \vdash A \wedge B$.
- ($\vee L$) Since $\mathfrak{R}(p)$ is prime, either $\mathfrak{R}(p) \vdash A$ or $\mathfrak{R}(p) \vdash B$ (Lemma 2.1, disjunction property). In either case, we can apply the i.h. to one of the two premises and obtain the desired result.

- $(\vee R)$ Applying the i.h. to the premise, we have that either there exists a $B'[q'] \in \Delta$ such that $\mathfrak{R}(q') \vdash B'$ (giving us the result) or else either $\mathfrak{R}(p) \vdash A$ or $\mathfrak{R}(p) \vdash B$, which implies $\mathfrak{R}(p) \vdash A \vee B$.

- $(\supset L)$ We apply the i.h. to the first premise, to obtain that either $\mathfrak{R}(pq) \vdash A$ or else there exists a $B'[q'] \in \Delta$ such that $\mathfrak{R}(q') \vdash B'$. In the latter case we are done, so assume the former. Now, since $\mathfrak{R}(p) \vdash A \supset B$ and $\mathfrak{R}(p) \subseteq \mathfrak{R}(pq)$, we have that $\mathfrak{R}(pq) \vdash A \supset B$ and hence $\mathfrak{R}(pq) \vdash B$ by modus ponens. But then we can apply the i.h. to the second premise and conclude that there is a $B'[q'] \in \Delta$ such that $\mathfrak{R}(q') \vdash B'$.

- $(\supset R)$ Let S be the set of prime extensions of $\mathfrak{R}(p), A$. For each $\Gamma' \in S$, we extend $\mathfrak{R}$ to a realization $\mathfrak{R}_{\Gamma'}$ by assigning $\mathfrak{R}_{\Gamma'}(pa) = \Gamma'$. We have that $\mathfrak{R}_{\Gamma'}(p') = \mathfrak{R}(p') \vdash A'[p']$ for all $A'[p'] \in \Gamma$, and moreover $\mathfrak{R}_{\Gamma'}(pa) \vdash A$, hence by the i.h. there must exist a hypothesis $B'[q'] \in B[pa], \Delta$ such that $\mathfrak{R}_{\Gamma'}(q') \vdash B'[q']$. Either $B'[q'] = B[pa]$, or else $B'[q'] \in \Delta$. If $B'[q'] = B[pa]$, then we have $\Gamma' \vdash B$. Otherwise, q' is distinct from pa by the freshness assumption, and $\mathfrak{R}_{\Gamma'}(q') = \mathfrak{R}(q') \vdash B'$. Therefore, either there is some $B'[q'] \in \Delta$ such that $\mathfrak{R}(q') \vdash B'$, or else for all $\Gamma' \in S$, $\Gamma' \vdash B'$. In the first case we are done. In the second case, since S is the set of prime extensions for $\mathfrak{R}(p), A$, we have that $\mathfrak{R}(p), A \vdash B$, whence $\mathfrak{R}(p) \vdash A \supset B$.

- $(\bot L)$ This case is impossible, since by assumption $\mathfrak{R}(p)$ is prime, which implies $\mathfrak{R}(p) \nvdash \bot$ (Lemma 2.1, consistency).

■

Corollary 3.2 (Soundness) *Suppose $\Gamma \vdash^{\ell} \Delta$, where every hypothesis in Γ and every conclusion in Δ occur at the same world p. Then $|\Gamma| \vdash \bigvee |\Delta|$, where $|-|$ erases the labels from a context.*

Proof: Let S be the set of prime extensions of $|\Gamma|$. For each $\Gamma' \in S$, we invoke the Realization Theorem with the realization $p \mapsto \Gamma'$. The theorem tells us that for each Γ', there exists a formula in $|\Delta|$ entailed by Γ'. These formulas are not necessarily the same for each Γ', however we have that always $\Gamma' \vdash \bigvee |\Delta|$. Since S is the set of prime extensions of $|\Gamma|$, this implies $|\Gamma| \vdash \bigvee |\Delta|$. ■