

CDM

Relations

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Battleplan

- Relations
- Operations on Relations
- Properties of Relations
- Order Relations
- Equivalence Relations
- Closures

Relations

Relations

We have seen how to express general concepts (or properties) as sets: we form the set of all objects that “fall under” the concept (in Frege’s terminology). Thus we can form the set of natural numbers, of primes, of reals, of continuous functions, of stacks, of syntactically correct C-programs and so on.

Another fundamental idea is to consider relationships between two or more objects. Here are some typical examples:

- divisibility relation on natural numbers,
- less-than relation on integers,
- greater-than relation on rational numbers,
- the “attends course” relation for students and courses,
- the “is prerequisite” relation for courses,
- the “is a parent of” relation for humans,
- the “terminates on input and produces output” relation for programs and inputs and outputs.

Binary Relations

Let's focus on the binary case where two objects are associated, though not necessarily from the same domain.

The unifying characteristic is that we have some property (attribute, quality) that can hold or fail to hold of any two objects from the appropriate domains.

Standard notation: for a relation P and suitable objects a and b write $P(a, b)$ for the assertion that P holds on a and b .

Thus we can associate a truth value with $P(a, b)$: the assertion holds if a and b are indeed related by P , and is false otherwise.

For example, if P denotes the divisibility relation on the integers then $P(3, 9)$ holds whereas $P(3, 10)$ is false.

Later we will be mostly interested in relations where we can effectively determine whether $P(a, b)$ holds, but for the time being we will consider the general case.

Implementation

Following our program of representing all important notions in terms of sets, here is a definition that expresses relations as sets.

Definition 1.

A **binary relation** ρ from A to B is a subset of $A \times B$.

A is the **domain** of ρ , and B is the **codomain** of ρ .

We write $\text{Rel}_{A,B}$ for the collection of all relations from A to B .

Notation

We will sometimes write $\rho : A \rightarrow B$ instead of $\rho \subseteq A \times B$.

Also, it is customary to use infix notation $x \rho y$ or prefix notation $\rho(x, y)$ instead of the set-theoretic $(x, y) \in \rho$.

For example, the standard order on the naturals will be indicated by $2 < 5$ rather than $<(2, 5)$.

Square Relations

An important special case arises when the domain and codomain are the same.

Definition 2.

A **square relation** (or an **endorelation**) is a binary relation with the same domain and codomain. The domain is called the **carrier set** or **underlying set** of the relation.

We write Rel_A for the collection of square relations on A .

On occasion, we will simply refer to these relations as binary relations on A .

Example 1. *The standard order relation $<$ on the natural numbers is a square relation.*

Typical counterexamples: the relation “ r is a root of polynomial $p(x)$ ” or “string s is an element of stack S ”.

Relations as Boolean-valued Functions

Our definition is the standard one, but note that we could also represent a relation by a map

$$A \times B \rightarrow \{\text{ff}, \text{tt}\}$$

that associates pairs of objects with truth values. In other words, we identify a relation with its characteristic function. This is somewhat closer to common usage in programming languages and also has the advantage that we can draw pictures, at least for reasonably small domains.

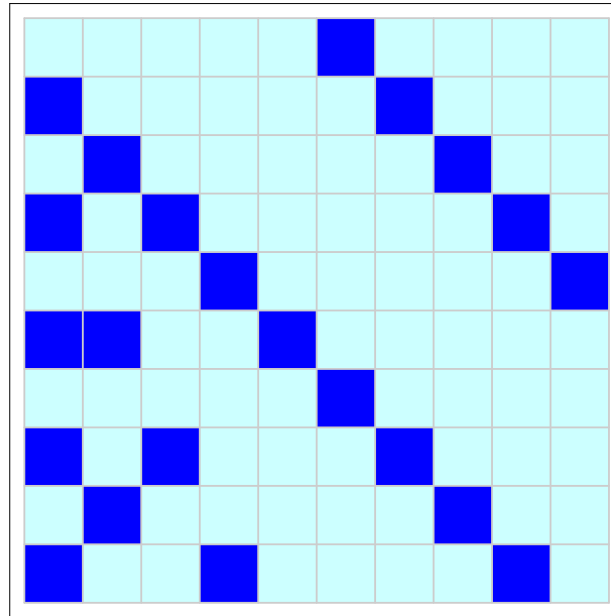
Suppose ρ is a relation on $[n]$ (or, more generally, a finite set of size n). We interpret ρ as a Boolean matrix, a 2-dimensional, n by n array $R : [n] \times [n] \rightarrow \mathbb{B}$ of Booleans such that

$$i \rho j \iff R[i, j] = \text{tt}.$$

On occasion, these picture give some insight into properties of the relation.

A Random Relation

Here is the picture for a “random” relation ρ on $[10]$.



Of course, ρ is far from random, in fact

$$x \rho y \iff x \neq y \wedge (y + 1 \mid x \vee x + 5 \mid y)$$

Relations as Set Functions

Another way to think about $\rho \subseteq A \times B$ is as a set-valued map

$$R : A \rightarrow \text{pow}(B)$$

that associates every object in the domain with a set of objects in the codomain in the obvious way:

$$R(a) = \{ b \in B \mid a \rho b \}.$$

E.g., for the last relation we obtain

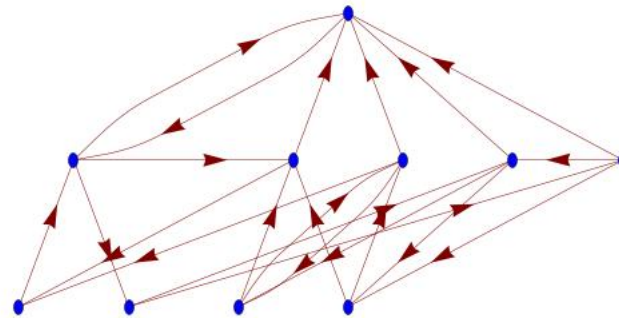
1	{6}	6	{1, 2, 5}
2	{1, 7}	7	{6}
3	{2, 8}	8	{1, 3, 7}
4	{1, 3, 9}	9	{2, 8}
5	{4, 10}	10	{1, 4, 9}

Visualization

This “array of lists” representation is very useful in a number of algorithms.

Moreover, it can be visualized by a “shoelace” picture where we plot A in the left column, B in the right and connect a with b by a line if $a \rho b$.

The following picture is for the relation ρ from above.



Graphs

Here is yet another way to think about a square relation $\rho \subseteq A \times A$.

Definition 3. *Directed Graphs*

A **directed graph** (or **digraph**) is a structure $G = \langle V, E \rangle$ where

- V is a set of **vertices** (or **nodes**, **points**)
- $E \subseteq V \times V$ is a set of **edges** (or **arcs**, **lines**)

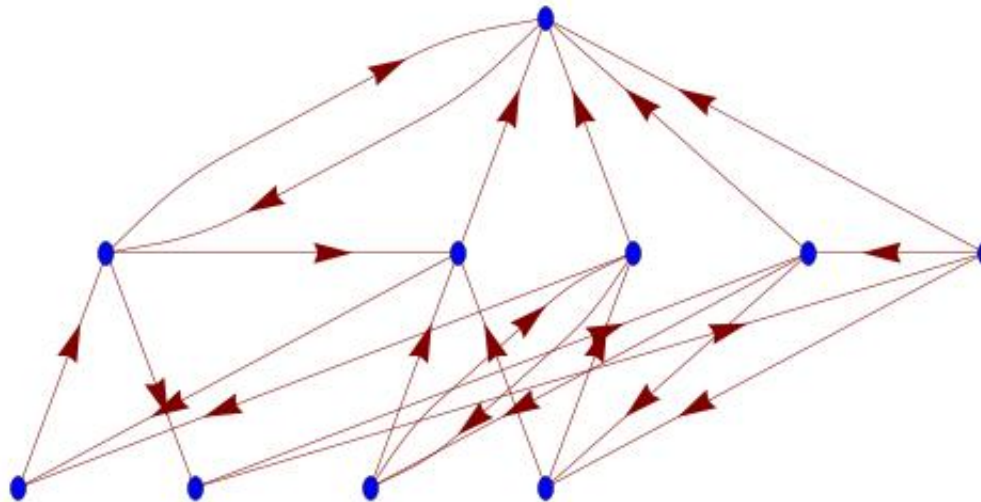
Edges are usually written as ordered pairs (u, v) : $u \in V$ is the **source**, and $v \in V$ the **target** of the edge. Of course, a digraph is logically the same as a binary relation, except that the carrier set has been spelled out explicitly.

► So why bother with graphs?

Because graph theory is an independent field of study, and historically has produced different results than the study of binary relations. Recall: Thinking about a problem just the right way is crucial in CS.

Graph Pictures

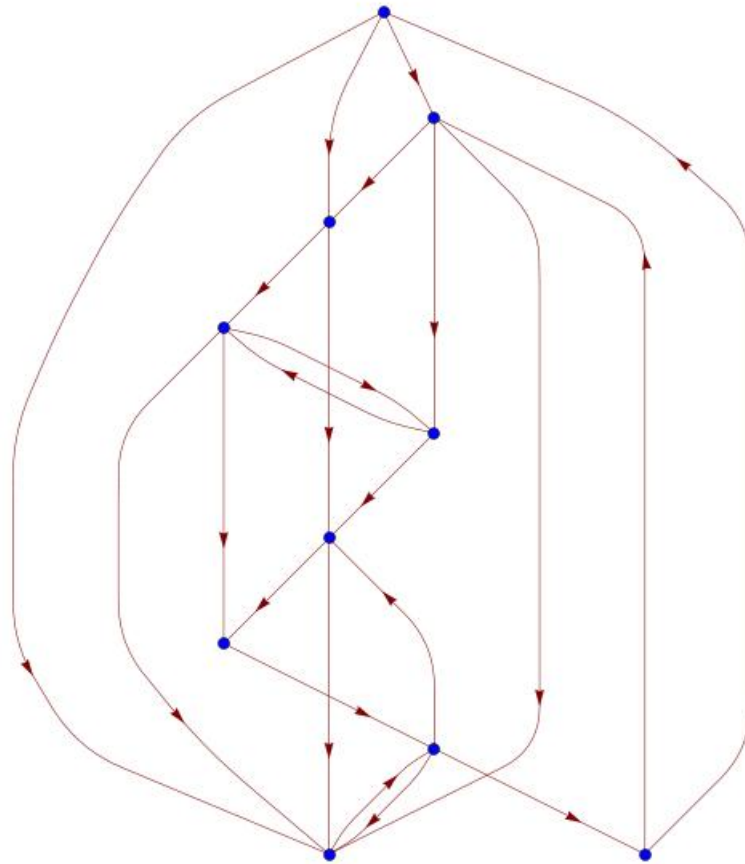
One interesting aspect of graph theory is that people have spent considerable time and effort on finding ways to represent graphs as sets of points in the plane or 3-space, with edges indicated by lines (or curves). In fact, there are some reasonably good graph layout programs which may produce useful visual representation of relations.



Note that this picture was generated automatically.

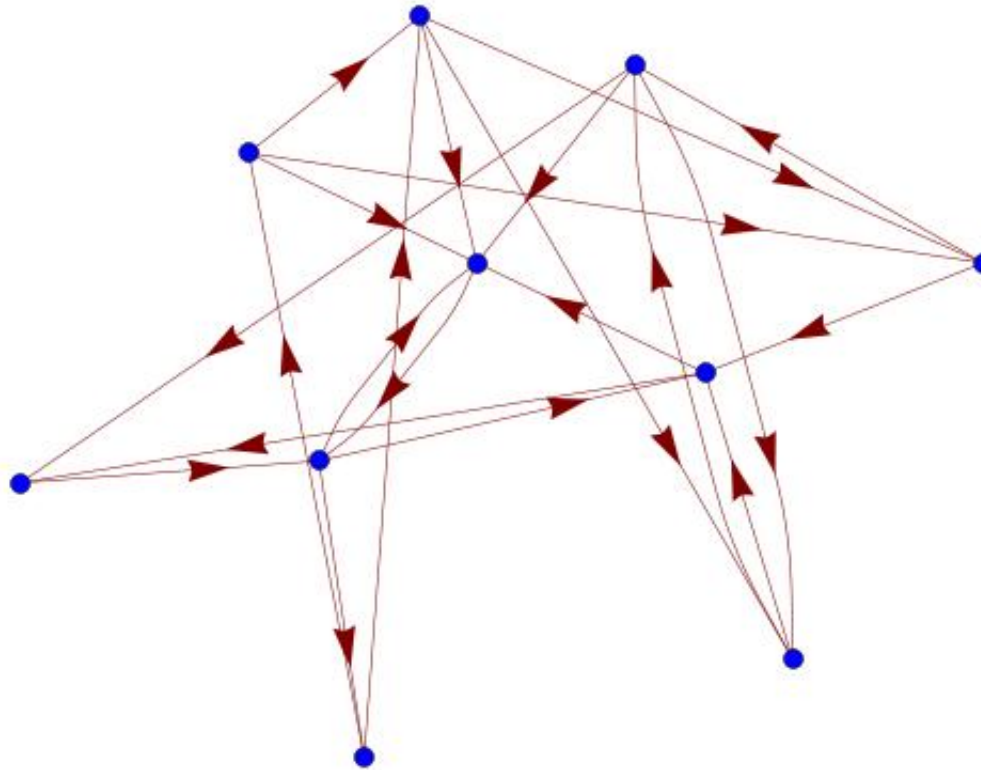
Caveat Emptor

Careful, though. The drawing represents the graph, but it is by no means canonical: there are many alternative ways to draw the same graph.



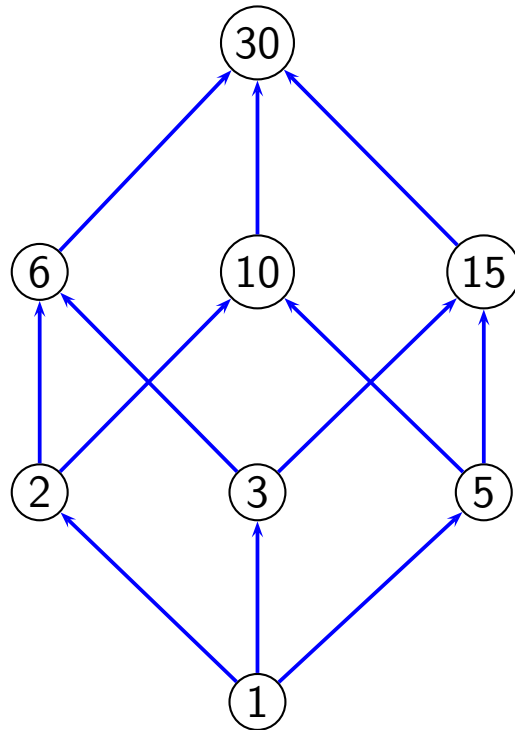
Yet Another . . .

Another picture for the same graph.



Divisor Lattice

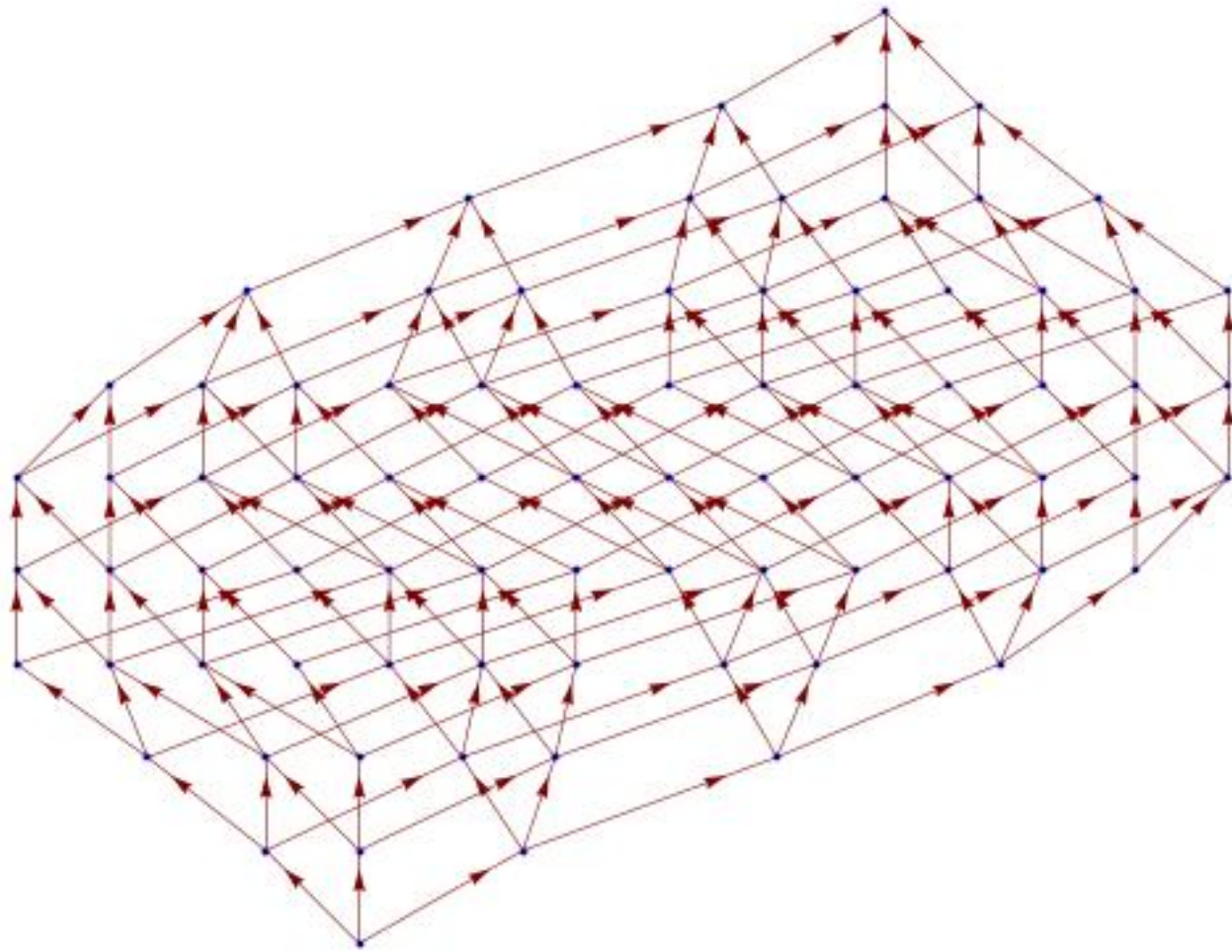
Careful selection of the vertex coordinates can help to convey a lot of information about the structure of the relation. The next picture was constructed painstakingly by hand.



This is the skeleton of divisibility on the divisors of 30.

And Automatic

Here is the automatically generated picture for the divisor lattice of 148176.



Special Relations

We focus on square relations on ground set A :

Definition 4.

- $I_A = \{ (x, x) \mid x \in A \}$, the **identity** or **diagonal** relation.
- $U_A = A \times A$, the **universal** relation.
- $\emptyset_A = \emptyset$, the **empty** relation.

Exercise 1. *What would the pictures for these relations look like in the various models above?*

Exercise 2. *Are there any other natural relations on A that one might reserve special notation for? Why?*

Counting Relations

Lemma 1. *If $|A| = n$ and $|B| = m$, then there are 2^{nm} relations from A to B .*

Proof. The collection of all relations from A to B is simply the powerset of $A \times B$.

The latter has size mn , and our claim follows.

□

Another way to think about this: we can represent each relation ρ by an n by m Boolean matrix. There are $n \cdot m$ bits, each can be on or off: 2^{nm} possibilities.

This result also holds in the infinite case (but requires a little cardinal arithmetic to state and prove).

Exercise 3. *Count relations based on the set-valued function representation.*

Operations and Properties

Comparing Relations

Since relations express properties (of pairs of objects) it is natural to consider logical connections between them.

Definition 5. *Let ρ and σ be two relations from A to B . ρ is finer than σ (or σ is coarser than ρ) if $x \rho y$ implies $x \sigma y$.*

In symbols:

$$\rho \sqsubseteq \sigma$$

If we think of relations as subsets of the Cartesian product $A \times B$ this simply says that $\rho \subseteq \sigma$. Of course, most relations are not comparable in this sense: neither one will be finer than the other.

Exercise 4. *Which of the knight's move relations should be expected to be finer than another? Look at the pictures, but don't rely on them.*

Boolean Operations on Relations

Definition 6. Let ρ and σ be two relations from A to B .

The **join** of ρ and σ is defined by $x (\rho \sqcup \sigma) y \iff x \rho y \vee x \sigma y$.

The **meet** of ρ and σ is defined by $x (\rho \sqcap \sigma) y \iff x \rho y \wedge x \sigma y$.

The **negation** or **complement** of ρ is defined by $x \rho^- y \iff \neg x \rho y$.

Note that negation is an involution: $\rho^{- -} = \rho$.

Example 2. Consider the usual order relations on \mathbb{N} . Then

$$\leq = < \sqcup I_{\mathbb{N}}$$

$$< = \leq \sqcap I_{\mathbb{N}}^-$$

Set Interpretation

If we think of relations from A to B as subsets of the Cartesian product the Boolean operations on relations translate into Boolean operations on sets.

$$\rho \sqcup \sigma = \rho \cup \sigma$$

$$\rho \sqcap \sigma = \rho \cap \sigma$$

$$\rho^- = A \times B - \rho$$

But note that this is a representation issue: if we were to implement the relations as digraphs or Boolean matrices things work out differently.

Exercise 5. Explain negation ρ^- in terms of \sqsubseteq .

Converse and Composition

Definition 7. The **converse** of a relation is defined by

$$x \rho^c y \iff y \rho x.$$

The converse is also written ρ^{-1} on occasion.

Note that converse is an involution: $(\rho^c)^c = \rho$. But don't confuse this with complement.

In the set-theoretic interpretation we simply reverse the pairs:

$$\rho^c = \{ (y, x) \mid (x, y) \in \rho \}$$

Exercise 6. Show that $(\rho \sqcap \sigma)^c = \rho^c \sqcap \sigma^c$.

Relational Composition

Definition 8. Suppose $\rho : A \rightarrow B$ and $\tau : B \rightarrow C$ are relations. The **(relational) composition** of ρ and τ is defined to be the relation $\sigma = \rho \bullet \tau : A \rightarrow C$ where

$$x \sigma y \iff \exists z \in B (x \rho z \wedge z \tau y).$$

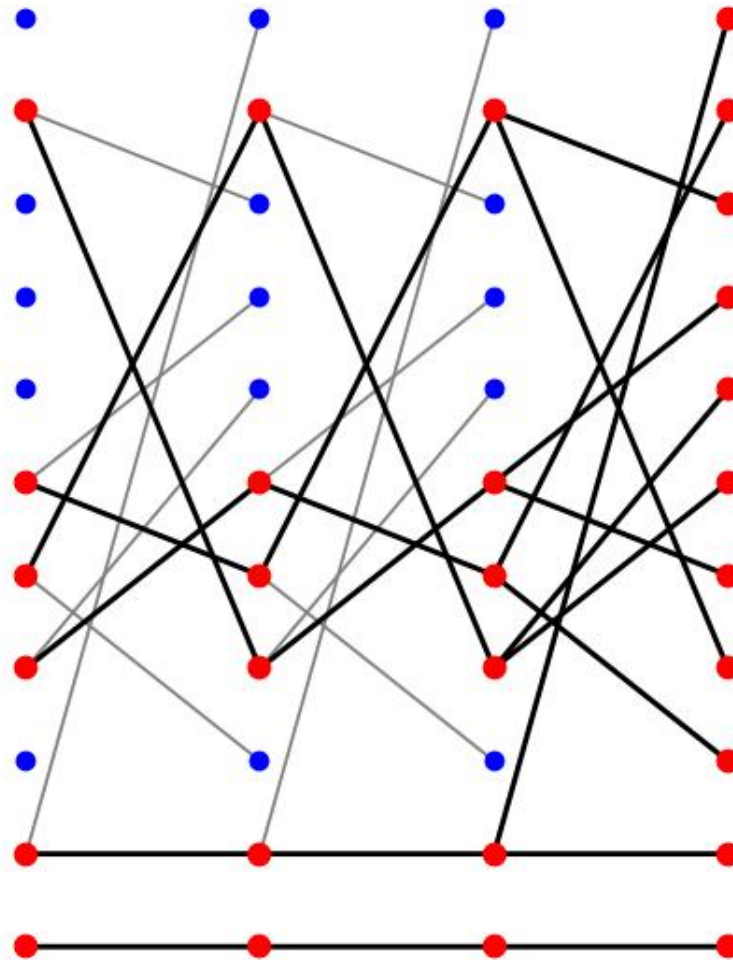
The intermediate element $z \in B$ is a **witness** (for $x \sigma y$).

Computationally composition is much more interesting than the previous operations. To test whether $x \sigma y$ holds we have to conduct a search for the witness z such that $x \rho z \tau y$.

Finding good algorithms to compute relational composition is a non-trivial problem; more later.

Visualizing Composition

Composing $x \rho y \iff x^2 = y \pmod{11}$.



Examples

Example 3. $\tau =$ “parent of”

$\tau \bullet \tau =$ “grandparent of”

$\tau^c \bullet \tau =$ “sibling of”

Example 4. $\tau =$ divisibility on \mathbb{N} : “ x divides y ”

Then $\tau \bullet \tau = \tau$

τ^c is “ x is a multiple of y ”

Example 5. $\tau =$ “less than” on \mathbb{N}

$\tau^c = >$, and $\tau \cup \tau^c = I_{\mathbb{N}}^-$ (inequality)

$\tau \bullet \tau \subset \tau$.

But for $A = \mathbb{Q}, \mathbb{R}$ we have $\tau \bullet \tau = \tau$.

Some Properties

Lemma 2. *Suppose $\rho : A \rightarrow B$, $\sigma : B \rightarrow C$, and $\tau : C \rightarrow D$ are relations.*

1. $\rho \bullet (\sigma \bullet \tau) = (\rho \bullet \sigma) \bullet \tau.$
2. $\rho \bullet I_B = I_A \bullet \rho = \rho.$
3. $\rho \bullet \emptyset = \emptyset \bullet \rho = \emptyset.$
4. $(\rho \bullet \sigma)^c = \sigma^c \bullet \rho^c.$

Proof. Tedious but rather straightforward application of definitions and a little logic. E.g., the first claim is established like so:

$$\begin{aligned}
 x (\rho \bullet (\sigma \bullet \tau)) y &\iff \\
 \exists u (x \rho u (\sigma \bullet \tau) y) &\iff \\
 \exists u, v (x \rho u \sigma v \tau y) &\iff \\
 \exists v (x (\rho \bullet \sigma) v \tau y) &\iff \\
 x ((\rho \bullet \sigma) \bullet \tau) y &
 \end{aligned}$$

□

YAM (Yet Another Monoid)

By the lemma, the set of all binary relations on A , with relational composition as operation, and the identity relation as neutral element, forms a monoid:

$$\langle \text{Rel}_A, \bullet, I_A \rangle$$

Hence one can study the structure of this monoid from an algebraic point of view. A more ambitious project would be to add other operations such as meet and join to the structure.

Exercise 7. *Analyze the structure of the monoid $\text{Rel}_{\{0,1\}}$.*

Exercise 8. *Is this monoid commutative? Is this monoid a group?*

Higher Powers

Definition 9. Let ρ be a relation on A . The **powers** ρ^i , $i \geq 0$, of ρ are defined by

$$\begin{aligned}\rho^0 &= I_A \\ \rho^{k+1} &= \rho \bullet \rho^k\end{aligned}$$

Composition on the left is arbitrary, we could also have chosen composition on the right.

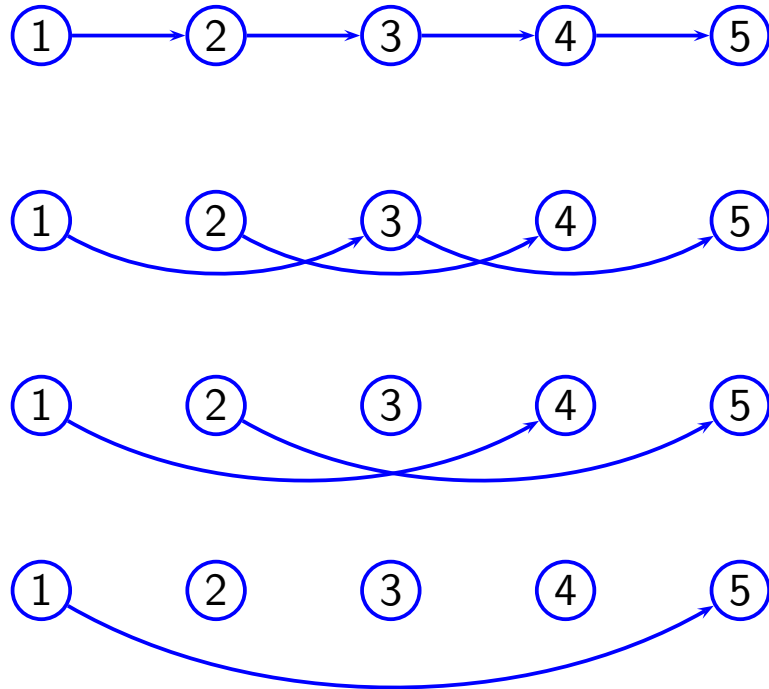
Thus $x \rho^k y$ if, and only if, there is a ρ -chain of length k from x to y :

$$\exists x_0, x_1, \dots, x_k (x = x_0 \rho x_1 \rho x_2 \rho \dots x_{k-1} \rho x_k = y)$$

In terms of the monoid of relations this is just exponentiation.

Power Example

The picture below shows the powers of the relation $x \rho y \iff y = x + 1$ on $[5]$. It is easy to see that in this case, $\rho^k = \emptyset$ for all $k \geq 5$, so only ρ through ρ^4 are shown.



Note that the powers of the relation are not comparable; in fact, in this particular case they are pairwise disjoint.

Chains

Let ρ be a square relation on A .

Definition 10. A ρ -chain is a sequence a_0, a_1, \dots, a_n of elements in A such that $a_i \rho a_{i+1}$ for all $i = 0, \dots, n - 1$. a_0 is the source of the chain, a_n its target and n its length.

This is often expressed informally in infix notation

$$a_0 \rho a_1 \rho \dots \rho a_{n-1} \rho a_n$$

Computationally, the most interesting problem related to chains is to check if there is a chain from some given source to a given target. This is mostly of interest for finite carrier sets where we have a reasonable representation of the relation.

Exercise 9. What is the longest chain in the divisibility relation (excluding equality) on a natural number n ?

Knights in Shiny Armor

Here is a situation where the Boolean matrix pictures are somewhat interesting.

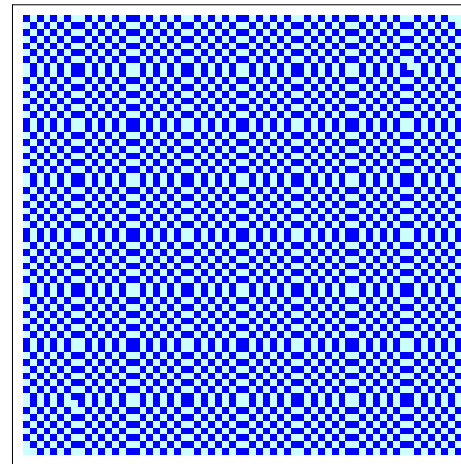
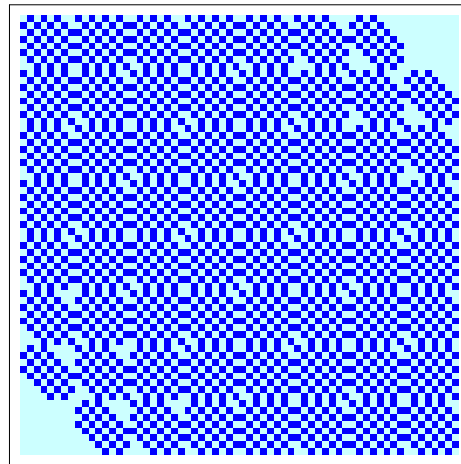
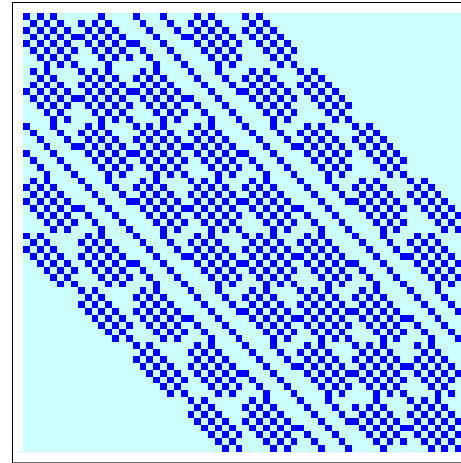
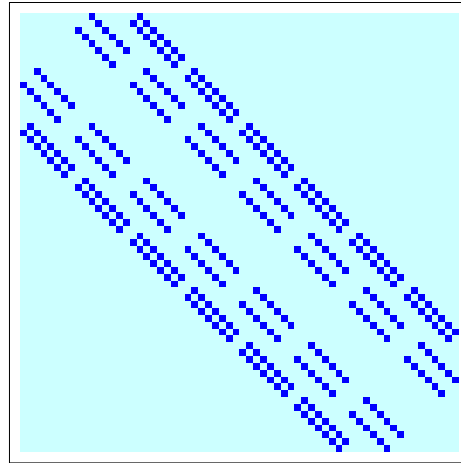
Consider the carrier set $A = [8] \times [8]$ which may be identified with the squares of a chess board. Define two squares to be related by ρ if a knight can move from one to the other.

Can a knight starting at square $(1, 1)$ reach all squares on a chessboard?

We can flatten the chessboard in row-major order so that the Boolean matrix representation of ρ will be 64 by 64.

The following pictures show reachability by a chain of knight moves, for lengths 1, 2, 3 and 4.

Knight Moves



Chains and Paths

Needless to say, chains are just the relation theoretic version of paths in graph theory.

Thus, if the relation models a communication network (direct links from station to station), a chain/path corresponds to the possibility of sending a message from one station to another (hopping).

There are standard algorithms in graph theory to tackle the chain/path existence problem, most notably depth-first-search and breadth-first-search.

Theorem 1. *The path existence problem in a finite digraph can be solved in time linear in the number of nodes plus the number of edges.*

Note that the space requirement is also linear in the size of the vertex set, though.

Basic Properties

There is a short list of basic properties of relations that can be combined to define important types of relations such as orders and equivalence relation.

Definition 11. *Let ρ be a binary relation on A .*

<i>property</i>	$\forall x, y, z \in A$
reflexive	$x \rho x$
irreflexive	$\neg(x \rho x)$
symmetric	$x \rho y \Rightarrow y \rho x$
asymmetric	$\neg(x \rho y \wedge y \rho x)$
antisymmetric	$x \rho y \wedge y \rho x \Rightarrow x = y$
transitive	$x \rho y \wedge y \rho z \Rightarrow x \rho z$

Exercise 10. *Give an example and a counterexample for each property.*

Relational Calculus

The logical definitions from above are standard, but one could also use the calculus of relations to pin down these properties.

property	condition
reflexive	$I \sqsubseteq \rho$
irreflexive	$I \sqcap \rho = \emptyset$
symmetric	$\rho^c \sqsubseteq \rho$
asymmetric	$\rho \sqcap \rho^c = \emptyset$
antisymmetric	$\rho \sqcap \rho^c \sqsubseteq I$
transitive	$\rho \cdot \rho \sqsubseteq \rho$

An interesting idea fact here is that this characterizations is entirely equational in terms of meet since we can express \sqsubseteq equationally as follows:

$$\rho \sqsubseteq \sigma \iff \rho \sqcap \sigma = \rho.$$

Examples

- equal-to, subset-of and divides are reflexive
- less-than, proper-subset-of and parent-of are irreflexive
- equal-to and relatively-prime are symmetric
- less-than and parent-of are asymmetric
- less-than-or-equal, subset-of and divides are antisymmetric
- equal-to, subset-of, divides and ancestor-of are transitive
- parent-of and relatively-prime are not transitive

Note that there are corresponding decision problems: how do we check whether a relation is, say, transitive?

At least for finite carrier sets one would like to have efficient algorithms.

Exercises

Exercise 11. Show that $(\rho \bullet \sigma)^c = \sigma^c \bullet \rho^c$.

Exercise 12. Show that $\rho \bullet \rho^c \bullet \rho \sqsubseteq \rho$ iff $\rho \bullet \rho^c \bullet \rho = \rho$.

Exercise 13. Suppose ρ is reflexive. Show that $\rho \bullet \rho^c \bullet \rho \sqsubseteq \rho$ iff ρ is also symmetric and transitive.

Exercise 14. Show that $\rho \bullet (\sigma \sqcup \tau) = \rho \bullet \sigma \sqcup \rho \bullet \tau$.

Exercise 15. Show that $\rho \bullet (\sigma \sqcap \tau) \sqsubseteq \rho \bullet \sigma \sqcup \rho \bullet \tau$. Find an example that demonstrates that equality does not hold in general.

Orders

Orders

The relations \leq (say, on \mathbb{N}) and \subseteq clearly are similar.

What are the crucial properties that make them similar?

Definition 12. *Let ρ be a relation on A .*

ρ is a **preorder** if it is both reflexive and transitive.

ρ is a **partial order** if it is a preorder and antisymmetric.

ρ is an **order** or **total order** if it is a partial order and all elements of A are comparable:

$$\forall x, y \in A (x \rho y \vee y \rho x).$$

Equivalently, ρ is total if $\rho \sqcup \rho^c = U_A$.

Strict versus Reflexive

Often it is more convenient to consider the *strict* (i.e. irreflexive) version of a preorder, order and so forth, obtained by setting

$$x \rho' y \iff x \neq y \wedge x \rho y.$$

Notation: \leq versus $<$, \subseteq versus \subset

Thus, a strict preorder is any irreflexive and transitive relation. Note that any such relation is automatically asymmetric. A strict total order has the additional *trichotomy* property: $\forall x, y \in A (x \rho y \vee x = y \vee y \rho x)$.

Also, the converse of an order is often useful: \geq and $>$.

Exercise 16. Show that the converse of a (strict) pre/partial/total order is again a (strict) pre/partial/total order.

Posets

Another piece of terminology: a structure

$$\langle A, < \rangle$$

consisting of a set A and a partial order $<$ (strict or not) is often called a *poset*.

This is mostly an acknowledgment that partial orders are so important that one should have a nice, compact, generally accepted name for them (just like groups or fields).

The study of posets has turned out to be of major importance for the semantics of programming languages.

Examples

- The usual order relations on numbers are total orders.
- Subset-of and divides are partial orders.
- Ordering polynomials by their degree produces a preorder.
- Cardinality is a preorder (and has full comparability).
- Lies-north-east-of in the plane is a partial order.
- Lexicographic order on words is a total order.
- The substring order is a partial order on strings:
$$x \preceq y \iff \exists u, v (y = uxv).$$
- The element relation \in is a total order on the class of ordinals.

Ordering Pairs

A standard problem is to lift a given order \leq on A to $A \times A$.

One plausible definition is

Definition 13. The **product order** of \leq with itself is defined by

$$(x, y) \leq_2 (x', y') \iff x \leq x' \wedge y \leq y'.$$

Example 6. On the plane $\mathbb{R} \times \mathbb{R}$ this is the “north-east-of” relation.

Thus, the product order is partial, even though $\langle \mathbb{R}; \leq \rangle$ is a total order. Can we manufacture a total order?

How about

$$(x, y) \trianglelefteq (x', y') \iff x \leq x' \vee (x = x' \wedge y \leq y').$$

Exercise 17. Show that \trianglelefteq is indeed a total order.

Ordering Sequences

It is slightly harder to order all sequences over A , not just pairs.

The usual order on characters $a < b < c < \dots < y < z$ can be lifted to words over this alphabet.

Recall that we write $|u|$ for the length of a sequence u .

Definition 14. Consider two sequences u and v over A .

1. *Lexicographic Order (Dictionary Order)*

$u \prec v$ **lexicographically** if u is a proper prefix of v or if $u = xay$, $v = xbz$ and $a < b$, where x, y, z are sequences, $a, b \in A$.

2. *Length-Lex Order*

$u \prec v$ in the **length-lex** order if $|u| < |v|$ or $|u| = |v|$ and u precedes v lexicographically.

3. *Length Order*

$u \prec v$ in the length order if $|u| < |v|$.

Lex Orderings

Lexicographic order and length-lex order are bonified total orders on finite sequences, but length order is only a preorder.

On words over the alphabet $\{0, 1\}$, length-lex order produces

$$\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \dots$$

This is crucially important for induction arguments: we finish off all words of length k before working on length $k + 1$.

But lexicographic order for these few words would be

$$\varepsilon, 0, 00, 000, 001, 010, 011, 1, 11$$

This is much less natural if one tries to systematically work through these words.

► Lexicographic order is used in the C++ library STL to overload operator< for stacks.

Well-Orders

Talking about induction, here is the key requirement for inductive arguments and constructions.

Definition 15. *A binary relation ρ on A is **well-founded** if every non-empty subset of A contains a ρ -minimal element. If ρ is also a strict total order it is **well-ordered**.*

So for a well-founded relation $\langle A, < \rangle$ we have

$$\emptyset \neq X \subseteq A \Rightarrow \exists a \in X \forall z \in A (z < a \rightarrow z \notin X).$$

Equivalently, there are no infinite descending chains in $<$:

$$a_0 > a_1 > a_2 > \dots > a_n > \dots$$

Inductive Properties

Definition 16. Let $\langle A, < \rangle$ be a well-founded relation. A subset $X \subseteq A$ is **inductive** if $\forall x < y (x \in X) \rightarrow y \in X$.

The fundamental result that enables us to perform induction is the following.

Theorem 2. Let $\langle A, < \rangle$ be well-founded and $X \subseteq A$ inductive. Then $X = A$.

So we can perform induction on the natural numbers with the standard order, on words in length-lex order, on ordinals with the standard order, on inductively defined data structures such as trees and lists, and so on.

Exercise 18. Show that we can also perform induction in a suitable partial order. In fact, not even transitivity is needed, just the well-ordering property from above.

Some Well-Founded Relations

The classic example of a well-ordering is $\langle \mathbb{N}, < \rangle$, the standard order on the natural numbers.

But other number systems such \mathbb{Z} , \mathbb{Q} and \mathbb{R} all fail to be well-founded. Do not confuse this with the Well-Ordering Principle which says that every set can be well-ordered: they can be, but the standard order does not do the job.

For computer science, recursive data-types produce important examples of well-founded relations. Many operations on data structures such as lists, words and trees can be explained in terms of induction along these well-founded relations.

Lemma 3. *The direct (pointwise) product of two well-founded relation is well-founded. The lexicographic product of two well-founded relation is well-founded.*

Exercise 19. *Prove the lemma. How about well-founded partial orders and well-orderings?*

Equivalence Relations

Generalizing Equality

Equality is one of the most important relations. Clearly, equality is reflexive, symmetric and transitive:

- $x = x$,
- $x = y$ implies $y = x$, and
- $x = y, y = z$ implies $x = z$.

It is natural to consider weaker versions of equality: coarser relations that maintain these three properties.

These relations formalize the notion of two objects being “the same” in some sense – without necessarily being identical.

Equivalence Relations

Definition 17. A relation ρ on A is an **equivalence relation** if ρ is symmetric, reflexive, and transitive.

Suppose ρ equivalence relation. The **equivalence class** of $a \in A$:

$$[a]_\rho = \{ x \in A \mid a \rho x \}.$$

$A/\rho = \{ [a]_\rho \mid a \in A \}$ is the collection of all such classes. The cardinality of A/ρ is the **index** of ρ .

Recall that for a binary relation ρ on some set A we have

property	$\forall x, y, z \in A$
<i>reflexive</i>	$x \rho x$
<i>symmetric</i>	$x \rho y \rightarrow y \rho x$
<i>transitive</i>	$x \rho y \wedge y \rho z \rightarrow x \rho z$

Examples

- equality is an equivalence relation
- same-weight is an equivalence relation on dumbbells
- same-parents is an equivalence relation on humans
- same-cardinality is an equivalence relation on sets
- same-area is an equivalence relation on polygons
- same-input-output is an equivalence relation on programs

Note that these are all of the form

$$x \rho y \iff f(x) = f(y)$$

for some suitable function f .

The Classic

Congruence modulo m on the integers.

$$x = y \pmod{m} \iff m \text{ divides } x - y$$

Is an equivalence relation on \mathbb{Z} where

$$[a] = \dots, a - 2m, a - m, a, a + m, a + 2m, \dots$$

Note that the equivalence relation \pmod{m} has index m .

This simple equivalence is the foundation for a lot of cryptographic schemes, including RSA.

Was first studied in great detail by C. F. Gauss (1777–1855) in his work on number theory.

Equivalence Classes

What can we say about equivalence classes?

The equivalence classes of I_A and U_A are trivial:

$$\{a\} = [a]_{=} \quad \text{and} \quad A = [a]_{U_A}$$

In general, we always have

$$a \in [a]_{\rho}$$

by reflexivity.

Usually write $[a]$ instead of $[a]_{\rho}$ unless there is any danger of confusion.

Disjoint Classes

Here is the single-most important property of equivalence classes.

Lemma. *Let ρ be an equivalence relation on A . For all $a, b \in A$:*

$$[a] = [b] \text{ or } [a] \cap [b] = \emptyset.$$

Proof.

If $c \in [a] \cap [b]$, then by symmetry for any $z \in [a]$:

$$z \rho a \rho c \rho b$$

Hence $z \in [b]$ by transitivity. But then $[a] \subseteq [b]$.

By a symmetric argument, $[b] \subseteq [a]$, done. □

Partitions

Definition. A **partition** of a set A is a set $P \subseteq \mathfrak{P}(A)$ of subsets of A such that

- $X \neq \emptyset$ for all $X \in P$,
- $\bigcup P = A$, and
- $X \neq Y \in P \rightarrow X \cap Y = \emptyset$.

The sets $X \in P$ are called the **blocks** of the partition.

This is really just different terminology, not a new idea: a block is nothing but an equivalence class.

By the lemma, the classes form a partition.

Same Idea

Lemma. *Equivalence relations correspond exactly to partitions.*

Proof.

If ρ is an equivalence relation on A the equivalence classes $[a]$ of ρ produce a partition by the last lemma.

Now let P be a partition of A . Define

$$x \rho y \iff \exists X \in P (x, y \in X).$$

Clearly ρ is reflexive and symmetric.

Transitivity follows from the lemma. □

Knights

Consider the chess board $C = [8] \times [8]$.

Define a relation ρ on C by: $x \rho y$ iff a knight can move from x to y (sequence of single moves).

Claim. ρ is an equivalence relation.

There is exactly one class: $[(1, 1)]_\rho = C$. Same for a rook, king, queen.

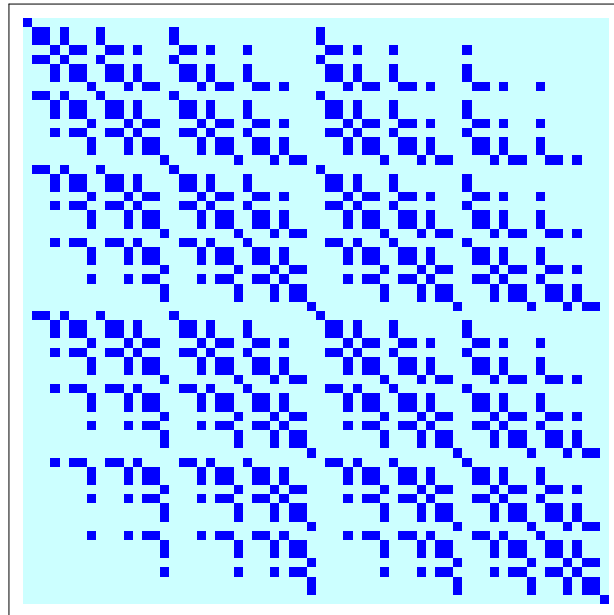
But for a bishop there are two equivalence classes: the black squares and the white squares.

Exercise 20. *How many equivalence classes for a pawn? Only consider the basic move $(i, j) \rightarrow (i, j + 1)$, no conversion. What's wrong?*

Boolean Matrices

Consider as carrier set the collection of all binary lists of length 6 (organized in natural order). Here is a picture of the relation “list x has the same number of 1’s as list y .”

Note that by definition this is an equivalence relation.

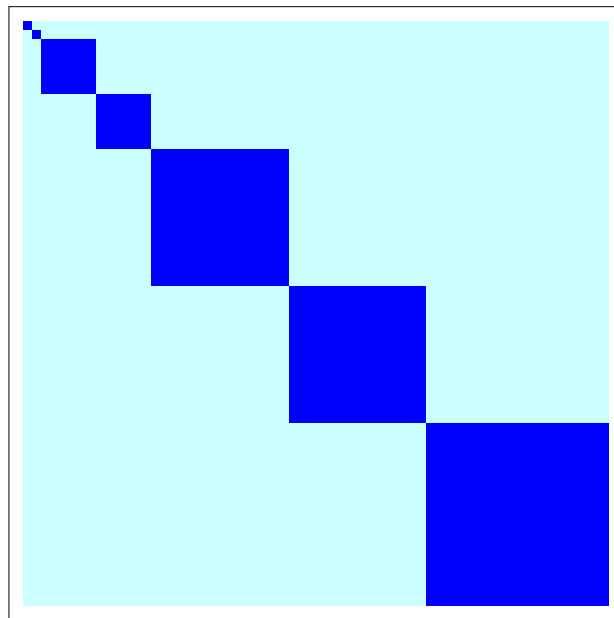


The picture clearly shows reflexivity and symmetry. But how about transitivity?

Reordering the Carrier Set

Transitivity is just about impossible to see from the last picture.

But, if we reorder the carrier set so that equivalent elements are all consecutive, the picture becomes very clear.



Exercise 21. *What are the sizes of these blocks?*

Reordering

Picture 1 uses the carrier set in natural order, but in picture 2 the carrier set is reordered so all elements of a block in the equivalence relation are contiguous.

natural	reordered
000000	000000
000001	111111
000010	000001
000011	000010
000100	000100
000101	001000
...	...
111101	110010
111110	110100
111111	111000

The important point here is that the relation is unchanged, only its representation in terms of the Boolean matrix has changed.

Equidecomposability

Here is a more interesting equivalence relation.

Definition 18. *Two polygons P and Q are equidecomposable if P can be cut up into finitely many triangles which can be reassembled to form Q .*

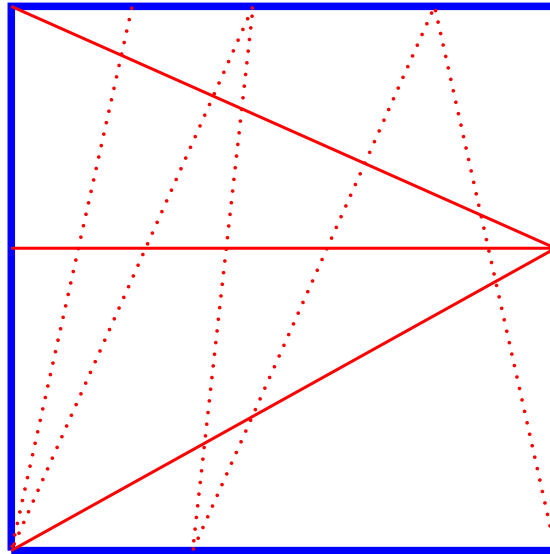
We will not give a precise definition of what we mean by reassemble and appeal to common (geometric) sense.

Lemma 4. *Equidecomposability is an equivalence relation.*

Proof. Reflexivity and symmetry is obvious. But transitivity is not: need the fact that the intersection of two polygons is another polygon (or a set of polygons) and that all polygons can be triangulated. \square

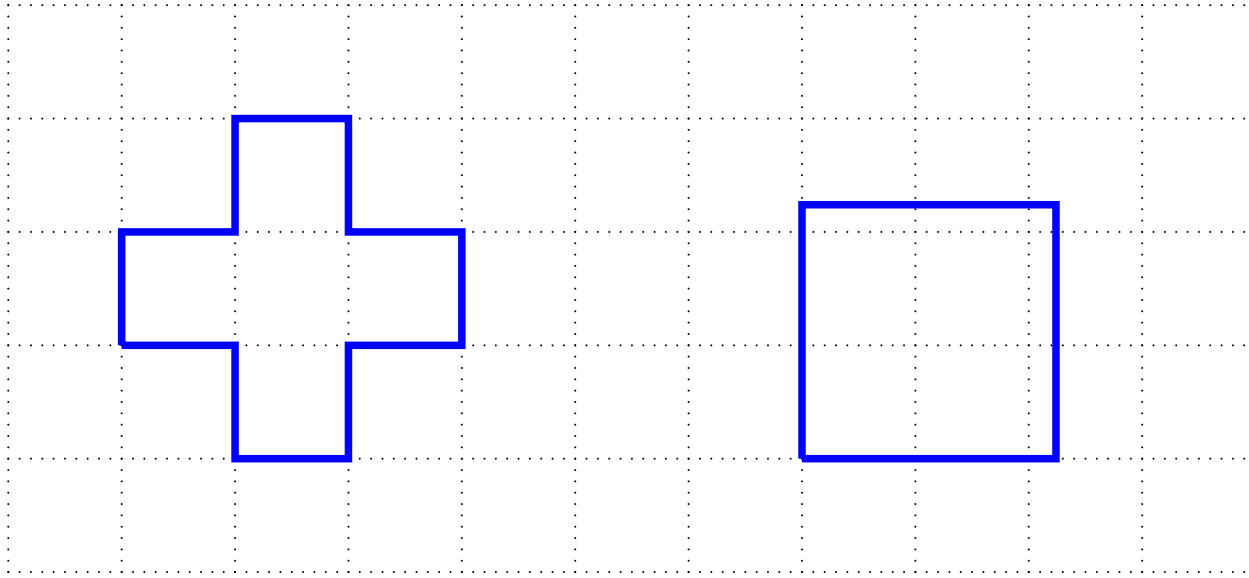
A Picture

Given two triangulations of the same polygon we can construct a finer one: just triangulate all the polygonal regions produced by the intersection of the two given triangulations.



Exercise 22. *Give a detailed proof of the lemma.*

Challenge



The square on the right is $\sqrt{5}$ by $\sqrt{5}$, so both polygons have area 5.

Exercise 23. *Show that the two polygons are equidecomposable. Use as few cuts as possible.*

The Bolyai-Gerwin Theorem

It is clear that any two equidecomposable polygons must have the same area: cutting them up and reassembling the pieces does not change the area.

But it is somewhat surprising that the opposite direction also holds.

Theorem. *Bolyai-Gerwin Theorem*

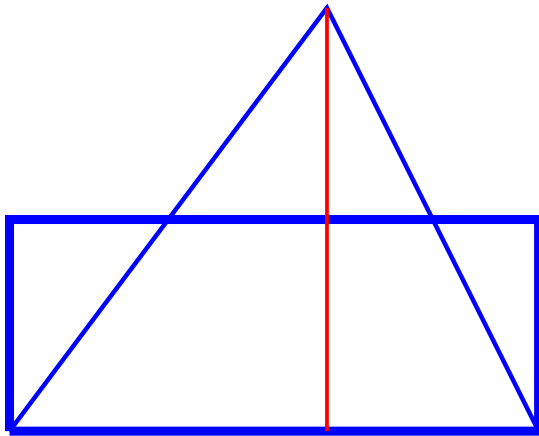
Let P and Q be two polygons of equal area. Then P and Q are equidecomposable.

We sketch the proof below. It is an excellent exercise to turn this into a real proof.

Step 1

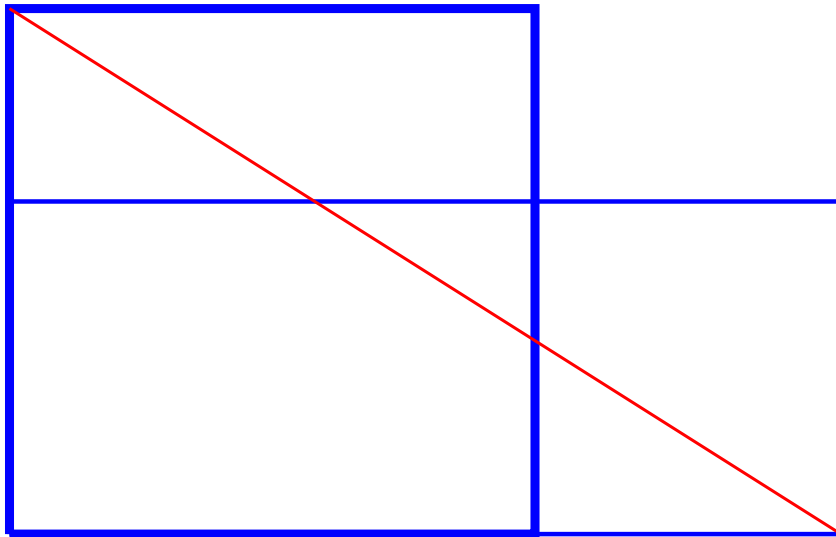
To prove the Bolyai-Gerwin theorem it suffices to show that every polygon P is equidecomposable with a square: there is exactly one square for each area. Since we can triangulate any polygon, we start with triangles.

Here is a method to convert triangles to rectangles of equal area:



Step 2

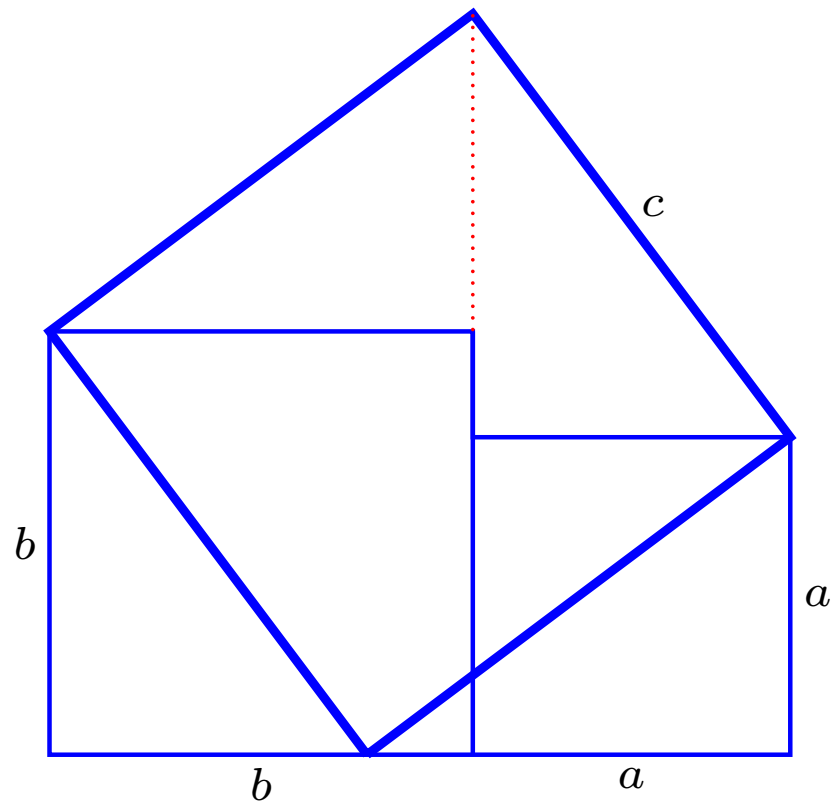
In the second step we convert rectangles to squares.



This requires that the longer side of the rectangles is ≤ 4 times the shorter one. What if not?

Step 3

Lastly, we have to add squares and produce new squares. We exploit Pythagoras' theorem:
 $a^2 + b^2 = c^2$.



Stroustrup, p. 467

Sorting algorithms in the STL can use orders supplied by user: provide a comparison operation $cmp(x, y)$. Must be a “strict weak ordering.”

```
sort( A.begin(), A.end(), cmp );
```

- [1] $cmp(x, x)$ is false.
- [2] If $cmp(x, y)$ and $cmp(y, z)$, then $cmp(x, z)$.
- [3] Define $equiv(x, y)$ to be $!(cmp(x, y) || cmp(y, x))$. If $equiv(x, y)$ and $equiv(y, z)$, then $equiv(x, z)$.

► What on earth does this really mean?

Let's write ρ for the relation cmp , and τ for $equiv$.

Note that ρ is transitive and irreflexive by [1] and [2].

Stroustrup, Explained

Claim. τ is an equivalence relation.

First off, $\tau = (\rho \cup \rho^c)^-$. Transitivity of τ is given.

Reflexivity follows from the fact that both ρ and ρ^c are irreflexive.

Symmetry follows from $\rho \cup \rho^c$ being symmetric, and symmetry being preserved by complementation. \square

So, the specification really says: incomparable elements have to be equivalent, we don't care which order they appear in.

Otherwise the sorting algorithm will not work.

- Sorting complex numbers according to their absolute value is fine.
- Sorting lists according to their length is fine.
- Sorting subsets according to cardinality will work.
- Sorting an array of subsets according to $x \subseteq y$ will not work.

Closures

Closure Operations on Relations

A frequent computational problem is to enlarge a given a relation ρ on A a bit until it has some property P .

Definition 19. *Let ρ be a relation on A and P a property of relations. The P -closure of ρ is the \sqsubseteq -least relation σ such that $\rho \sqsubseteq \sigma$ and σ has property P .*

Important cases are: reflexive closure, symmetric closure, transitive closure; combinations thereof.

Notation:

The transitive reflexive closure is often written ρ^* . The transitive closure is written ρ^+ .

► How do we know that the, say, transitive closure exists? We have to argue that a relation with certain properties actually exists.

Closures as Intersections

Claim 1. *If $\rho \sqsubseteq \sigma_i$ and σ_i is transitive, $i = 1, 2$, then $\sigma_1 \sqcap \sigma_2$ is again transitive.*

This claim is true even for an infinite family $(\sigma_i)_{i \in I}$.

But then, abstractly, the closure is

$$\sigma = \bigcap \{ \tau \sqsubseteq A \times A \mid \rho \sqsubseteq \tau \wedge \tau \text{ has property } P \}$$

The set on the right can never be empty: U_A is in there.

Of course, this is useless for computational purposes: we need algorithms, not general abstract nonsense.

A fine point: set on the right contains the relation whose existence we worry about (impredicative definition).

Other Closures

In the same way we can define *reflexive closure*, *symmetric closure*, *transitive closure* and *reflexive transitive closure*.

Notation:

$$\text{eqcl}(\rho), \text{rcl}(\rho), \text{scl}(\rho), \text{tcl}(\rho), \text{rtcl}(\rho)$$

Lemma 5. *All these operations F are idempotent: $F(F(\rho)) = F(\rho)$.*

Lemma 6.

- $\text{scl}(\text{rcl}(\rho)) = \text{rcl}(\text{scl}(\rho))$
- $\text{rtcl}(\rho) = \text{tcl}(\text{rcl}(\rho)) = \text{rcl}(\text{tcl}(\rho))$
- $\text{eqcl}(\rho) = \text{tcl}(\text{scl}(\text{rcl}(\rho)))$

Reflexive and Symmetric Closures

One can think of ρ as a succinct representation of σ : we have ρ , and we want to compute σ . Sometimes, that's easy.

Write $\text{rcl}(\rho)$ and $\text{scl}(\rho)$ for reflexive and symmetric closure, and $\text{rscl}(\rho)$ for the reflexive symmetric closure.

$$\text{rcl}(\rho) = \rho \sqcup I_A$$

$$\text{scl}(\rho) = \rho \sqcup \rho^c$$

$$\text{rscl}(\rho) = \rho \sqcup \rho^c \sqcup I_A$$

Example 7. *On the natural numbers we have: $\text{rcl}(<) = \leq$, $\text{scl}(<) = \neq = U_A - I_A$, $\text{rscl}(<) = U_{\mathbb{N}}$*

Exercise 24. *Show that $\text{rscl}(\rho) = \text{rcl}(\text{scl}(\rho)) = \text{scl}(\text{rcl}(\rho))$. Show that $\text{rcl}(\text{rcl}(\rho)) = \text{rcl}(\rho)$.*

Saving Space

Here is another example: equivalence as closure.

Suppose you need to implement equivalence relations on $[n]$ where n is, say, about 1000. There could be as many as 10^6 pairs that are related. Clearly, we do not wish to store so much information.

► What do we really need to know?

The property “is an equivalence relation” is yet another closure property. Get another closure operation: $ec(\rho)$:

$$ec(\rho) = tc(rsc(\rho)) = (I \sqcup \rho \sqcup \rho^c)^+$$

But the size of ρ may well be $O(n)$, while the size of $ec(\rho)$ is $O(n^2)$.

Powers and Transitive Closure

Powers and transitive closure are very closely connected. We could have defined the closures like so:

$$\text{rtcl}(\rho) = \bigsqcup_{k \geq 0} \rho^k = I_A \sqcup \rho \sqcup \rho^2 \sqcup \rho^3 \sqcup \dots$$

$$\text{tcl}(\rho) = \bigsqcup_{k > 0} \rho^k = \rho \sqcup \rho^2 \sqcup \rho^3 \sqcup \dots$$

Of course, this definition is a bit intimidating since it uses an infinitary operator.

But note that unlike with infinite series in calculus there is no problem with convergence here: no matter what the individual terms are, the big join will always exist.

Arbitrary Chains

This characterization simply says that $x \text{ tcl}(\rho) y$ iff

$$\exists k \geq 0 \exists x_0, x_1, \dots, x_k (x = x_0 \rho x_1 \rho \dots \rho x_{k-1} \rho x_k = y)$$

Note that length of the chains is actually bounded since A is finite.

Exercise 25. *One can always assume $k \leq n - 1$ where $n = |A|$.*

We will exploit this fact in the section on algorithms for relations.

Aside: k -ary Relations

We have limited our discussion to binary relations, but of course we can also study relations on k objects where $k \geq 1$. We can define relations in general to be subsets

$$\rho \subseteq A_1 \times A_2 \times \dots \times A_k$$

of general Cartesian products.

The case $k = 1$ is somewhat uninteresting, since unary relations are essentially the same as sets. However, k -ary relations for $k > 2$ do appear, in particular when dealing with databases.

For example, we might be interested in a relation on the product of

- A_1 : students
- A_2 : departments
- A_3 : degree programs
- A_4 : courses
- A_5 : grades
- A_6 : status indicator

We won't pursue this matter here, google for relational data bases.