15-451
Algorithms

Theory of NP-Completeness

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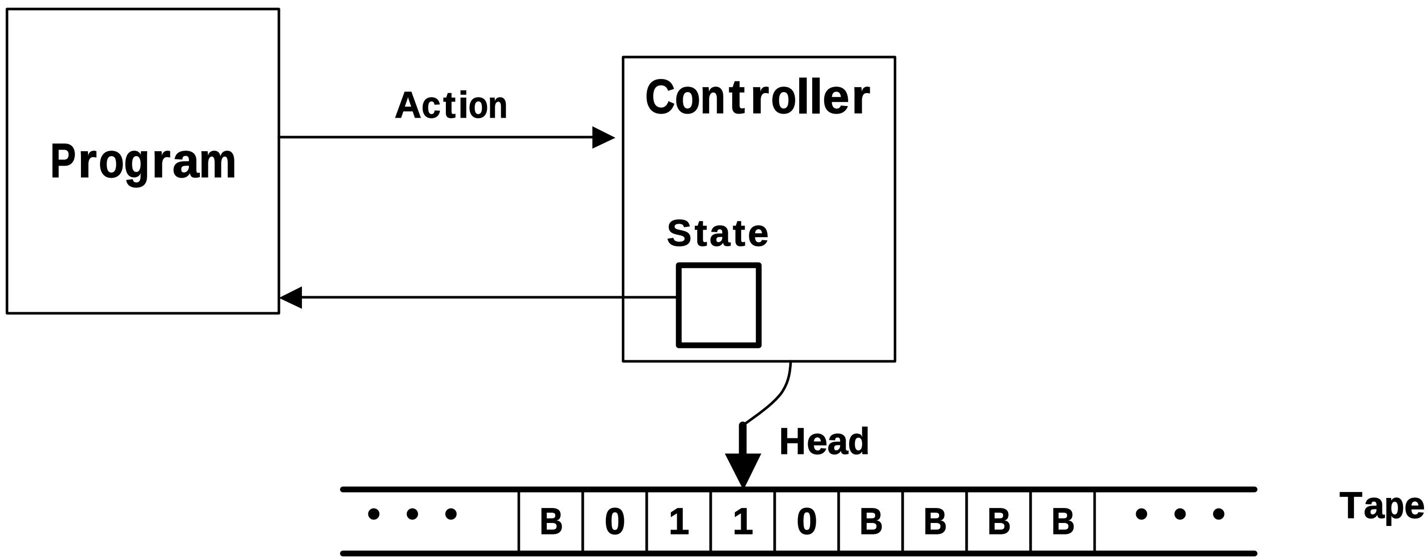
Topics:

- Turing Machines
- Cook's Theorem
- Implications

Turing Machine

Formal Model of Computer

- Very primitive, but computationally complete



Turing Machine Components

Tape

- **Conceptually infinite number of "squares" in both directions**
- **Each square holds one "symbol"**
  - From a finite alphabet
- **Initially holds input + copies of blank symbol 'B'**

Tape Head

- **On each step**
  - Read current symbol
  - Write new symbol
  - Move Left or Right one position

Components (Cont.)

Controller

- **Has state between 0 and m-1**
  - Initial state = 0
  - Accepting state = m-1
- **Performs steps**
  - Read symbol
  - Write new symbol
  - Move head left or right

Program

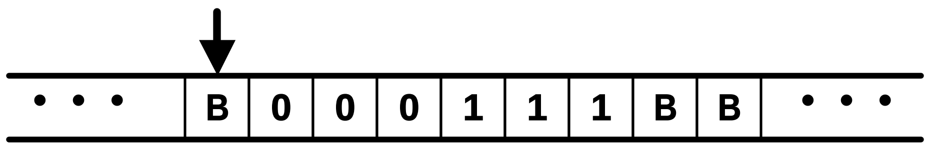
- **Set of allowed controller actions**
- **Current State, Read Symbol ® New State, Write Symbol, L|R**

Turing Machine Program Example

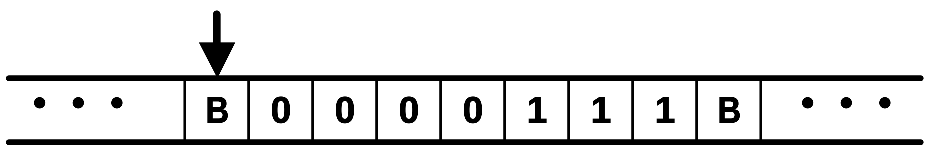
Language Recognition

- Determine whether input is string of form 0^n1^n

Input Examples



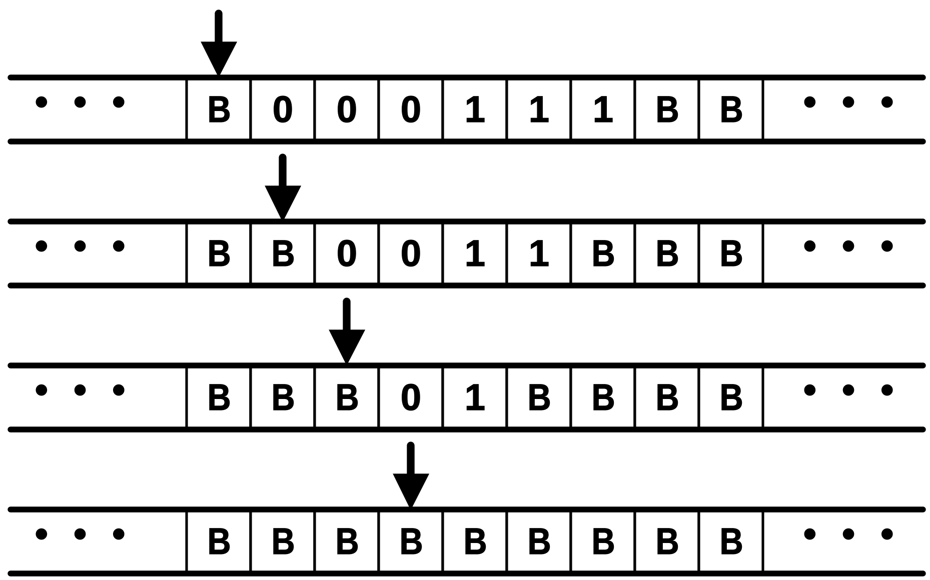
- Should reach state m-1



- Should never reach state m-1

Algorithm

- Keep erasing 0 on left and 1 on right
- Terminate and accept when have blank tape



Program

States

- 0 Initial
- 1 Check Left
- 2 Scan Right
- 3 Check Right
- 4 Scan Left
- 5 Accept

Read Symbol

Current State	B	0	1
0	1,B,R	–	–
1	5,B,R	2,B,R	–
2	3,B,L	2,0,R	2,1,R
3	–	–	4,B,L
4	1,B,R	4,0,L	4,1,L
5	–	–	–

'–' means no possible action from this point

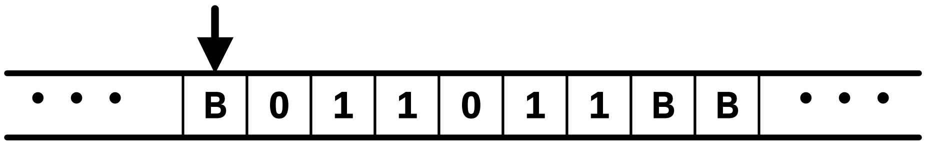
Deterministic TM: At most one possible action at any point

Non Deterministic Turing Machine

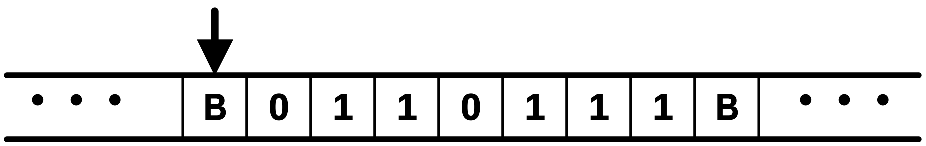
Language Recognition

- Determine whether input is string of form *xx*
- For some string x Î {0,1}*

Input Examples

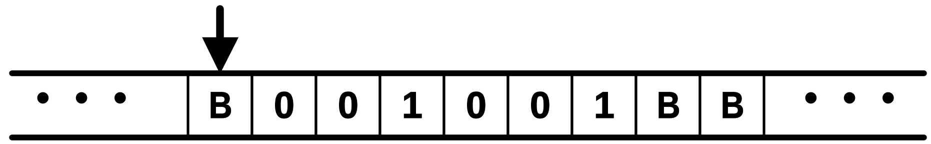


- Should reach state m-1

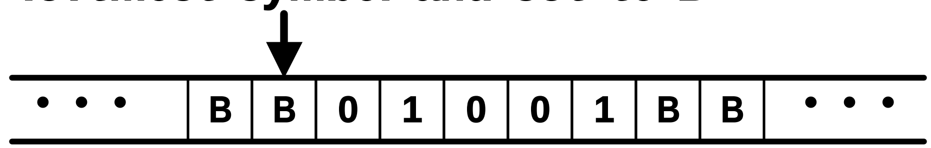


- Should never reach state m-1

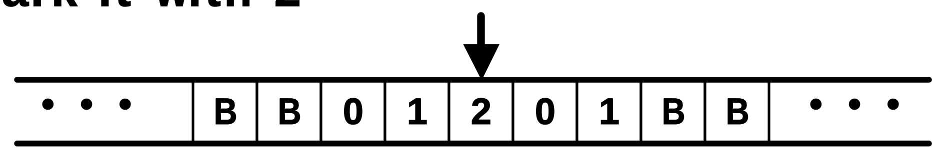
Nondeterministic Algorithm



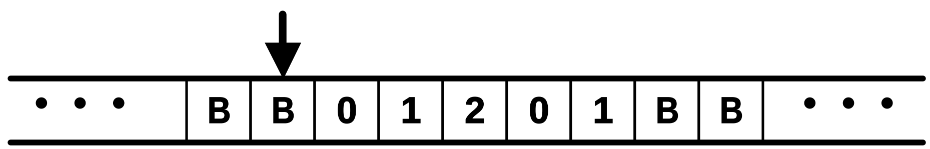
- **Record leftmost symbol and set to B**



- **Scan right, stopping at arbitrary position with matching symbol, and mark it with 2**

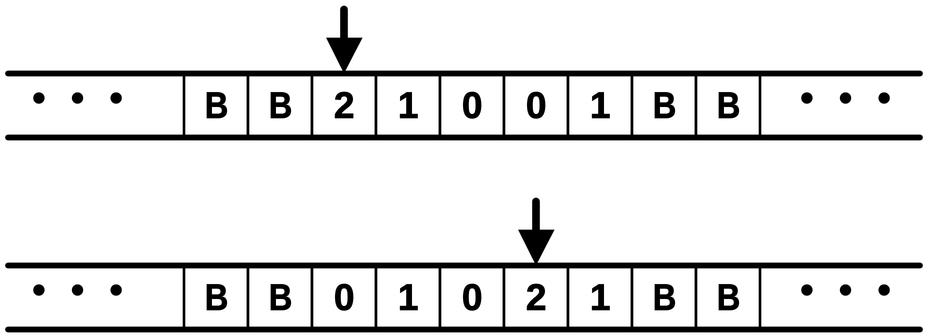


- **Scan left to end, and run program to recognize $x2^+x$**



Nondeterministic Algorithm

- **Might make bad guess**



- **Program will never reach accepting state**

Rule

- **String accepted as long as reach accepting state for *some* sequence of steps**

Nondeterministic Program

States

- 0 Initial
- 1 Record
- 2 Look for 0
- 3 Look for 1
- 4 Scan Left
- 5+ Rest of program

Read Symbol

Current State	B	0	1
0	1,B,R	–	–
1	accept,B,R	2,B,R	3,B,R
2	–	2,0,R 4,2,L	2,1,R
3	–	3,0,R	3,1,R 4,2,L
4	5,B,R	4,0,L	4,1,L

Nondeterministic TM: $\exists$ **2 possible actions from single point**

Turing Machine Complexity

Machine M Recognizes Input String x

- **Initialize tape to x**
- **Consider all possible execution sequences**
- **Accept in time t if can reach accepting state in t steps**
  - t(x): Length of shortest accepting sequence for input x

Language of Machine L(M)

- **Set of all strings that machine accepts**
- **x Ï L when no execution sequence reaches accepting state**
  - Might hit dead end
  - Might run forever

Time Complexity

- **T_M(n) = Max { t(x) | x Î L |x| = n }**
  - Where |x| is length of string x

P and NP

Language L is in P

- **There is some *deterministic* TM M**
  - $L(M) = L$
  - $T_M(n) = p(n)$ for some polynomial function p

Language L is in NP

- **There is some *nondeterministic* TM M**
  - $L(M) = L$
  - $T_M(n) = p(n)$ for some polynomial function p
- **Any problem that can be solved by intelligent guessing**

Example: Boolean Satisfiability

Problem

- **Variables:** $x_1, \ldots, x_k$
- **Literal:** either x_i or $\neg x_i$
- **Clause:** Set of literals
- **Formula:** Set of clauses
- **Example:** $\{x_3, \neg x_3\}$ $\{x_1, x_2\}$ $\{\neg x_2, x_3\}$ $\{x_1, \neg x_3\}$
  - Denotes Boolean formula $x_3 \vee \neg x_3 \;\wedge\; x_1 \vee x_2 \;\wedge\; \neg x_2 \vee x_3 \;\wedge\; x_1 \vee \neg x_3$

Encoding Boolean Formula

Represent each clause as string of 2k 0's and 1's

- 1 bit for each possible literal
- First bit: variable, Second bit: Negation of variable
- $\{x_3, \neg x_3\}$: 000011 $\{x_1, x_2\}$: 101000
- $\{\neg x_2, x_3\}$: 000110 $\{x_1, \neg x_3\}$: 100001

Represent formula as clause strings separated by '$'

- 000011$101000$000110$100001

SAT is NP

Claim

- **There is a NDTM M such that L(M) = encodings of all satisfiable Boolean formulas**

Algorithm

- **Phase 1: Determine k and generate some string {01,10}**
  - Append to end of formula
  - This will be a guess at satisfying assignment
  - E.g., 000011$101000$000110$100001*$100110*
- **Phase 2: Check each clause for matching 1**
  - E.g., 0000$\underline{1}$1\$$\underline{1}$01000\$0001$\underline{1}$0\$$\underline{1}$00001*$100110*

SAT is NP-complete

Cook's Theorem

- **Can generate Boolean formula that checks whether NDTM accepts string in polynomial time**

Translation Procedure

- **Given**
  - NDTM M
  - Polynomial function p
  - Input string x

- **Generate formula F**
  - F is satisfiable iff M accepts x in time p(|x|)
- **Size of F is polynomial in |x|**
- **Procedure generates F in (deterministic) time polynomial in |x|**

Construction

Parameters

- **|x| = n**
- **m states**
- **v tape symbols (including B)**

Formula Variables

- **Q[i,k]   0 = i = p(n), 0 = k = m-1**
  - At time i, M is in state k
- **H[i,j]   0 = i = p(n), -p(n) = j = p(n)**
  - At time i, tape head is over square j
- **S[i,j,k]   0 = i = p(n), -p(n) = j = p(n), 1 = k = v**
  - At time i, tape square j holds symbol k

Key Observation

- **For bounded computation, can only visit bounded number of squares**

Clause Groups

- Formula clauses divided into "clause groups"

Uniqueness

- At each time i, M is in exactly one state
- At each time i, tape head over exactly one square
- At each time i, each square j contains exactly one symbol

Initialization

- At time 0, tape encodes input x, head in position 0, controller in state 0

Accepting

- At some time i, state = m-1

Legal Computation

- Tape/Head/Controller configuration at each time i+1 follows from that at time i according to some legal action

Implications of Cook's Theorem

Suppose There Were an Efficient Algorithm for Boolean Satisfiability

- Then could take any problem in NP, convert it to Boolean formula and solve it quickly!
- Many "hard" problems would suddenly be easy

Big Question P =? NP

- Formulated in 1971
- Still not solved
- Most believe not

Complements of Problems

Language Complement

- **Define ~L = { x | x $\notin$ L}**
- **E.g., ~SAT**
  - Malformed formulas (easy to detect)
  - Unsatisfiable formulas

P Closed Under Complementation

- **If L is in P, then so is ~L**
  - Run TM for L on input x for p(|x|) steps
    - » Has unique computation sequence
  - If haven't reached accepting state by then, then x $\notin$ L

NP vs. co-NP (cont.)

Is NP = co-NP?

- **Having NDTM for ~L doesn't help for recognizing L**
  - Would have to check all computation sequences of length = p(|x|).
  - Could have exponentially many sequences

Proper Terminology

- **Generally want algorithm that can terminate with "yes" or "no" answer to decision problem**
- **If underlying problem (or its complement) is NP, then full decision problem is "NP-Hard"**

Showing Problems NP-Complete

To show Problem X is NP-complete

1. Show X is in NP

- **Can be solved by "guess and check"**
- **Generally easy part**

2. Show known NP-complete problem Y can be reduced to X

- **Devise translation procedure**
- **Given arbitrary instance y of Y, can generate problem x in X such that $y \in L_Y$ iff $x \in L_X$**
  - L_x: set of all strings x for which decision problem answer is "yes"
- **Size of x must be polynomial in y, and must be generated by (deterministic) polynomial algorithm.**