Symmetry Groups in Robotic Assembly Task Planning and Specification

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ABSTRACT

Group theory is the mathematical theory of symmetry. This book is trying to show that group theory provides a sound theoretical foundation for applications involving solid surface contact by allowing general, concise expressions describing the relative positions between solids. This formalism presents challenging computational problems due to the inhomogeneous nature of the symmetry groups that must be manipulated, ranging from finite and infinite discrete subgroups to continuous subgroups of the Euclidean group. This book presents a group theoretical formalization of surface contact among solids, together with a geometric representation and efficient group intersection algorithm for the TR subgroups of the Euclidean group. The relevence of symmetry group concepts and the effectiveness of the intersection algorithm are illustrated through a detailed design of an automated system for robotics assembly task specification and planning. Besides serving as an example for how technology can be enhanced by introducing a computational mathematical formalization, this book also serves as a step-by-step introduction to group theory for engineers.

Contents

Chapter 1

Group Theory and Contact

Tyger, Tyger burning bright,
In the forests of the night:
What immortal hand or eye,
Dare frame thy fearful symmetry?

William Blake

One basic question in robotics automation is **how to describe contacts between solids to a robot**? For example, how would you ask a robot to put a cube in a corner? This seemingly simple task requires 24 equivalent, but different, sets of task specifications if you wish to enumerate all the geometric possibilities. What if there is an infinite number of possible relative positions between the contacting solids such as in the case of putting a cylindrical peg in a cylindrical hole? It is a non-trivial task to communicate the *full* spatial relationships between locally symmetrical objects with a robot that does not *understand* symmetry. Such task specifications are forced to be either tedious and redundant, or suffering from incompleteness. Current engineering practice is still limited to a finite set of case-based scenarios.

Computing the relative positions of solids that are in contact is a fundamental problem in many fields, including robotics, computer graphics, computer aided design and manufacturing (CAD & CAM) and computer vision. It is the focus of this book to formalize solid contact based on local symmetry, to construct a computational framework using group theory, and to demonstrate the effectiveness of applications of computational group theory in robotics. Under our unified formalism, asymmetry is seen to be special cases of symmetry, contrary to today's common engineering practice where local or global symmetries of an object are often treated as degenerate or pathological instances from the norm.

1.1 Symmetry and Motion

Symmetries have long been used to describe the visual *appearance* of an object, a scene, or a set of symbols (music, language etc.). Figure 1.1 demonstrates the appearance of five regular solids. However using symmetry as a pertinent descriptor of the *functions* of an object is less explored. For example, why are car wheels round instead of square or any other shape with finite symmetries? Why do Philip screw drivers have a 4-fold symmetry at the tip? In particular, there is a tight coupling between the shape symmetry of *contacting surfaces* of two solids and the possible relative motions between the solids in contact. This coupling can be observed in Figure 1.2 where the six **lower pairs**[1] are shown.

1.1.1 Symmetry and Symmetry Groups

A simple abstract view of our 3D physical world contains two key basic components:

- an **object** S — a subset of R^3, and
- an **action, mapping**, or **transformation** g that can act on S in R^3.

Formally, we define symmetry as follows:

Definition 1.1.1 *A mapping* $g : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ *is a* **symmetry** *of* $S \subset \mathbf{R}^3$ *iff*

[1]Mechanical joints that have surface contact are called *lower pairs* in mechanical engineering.

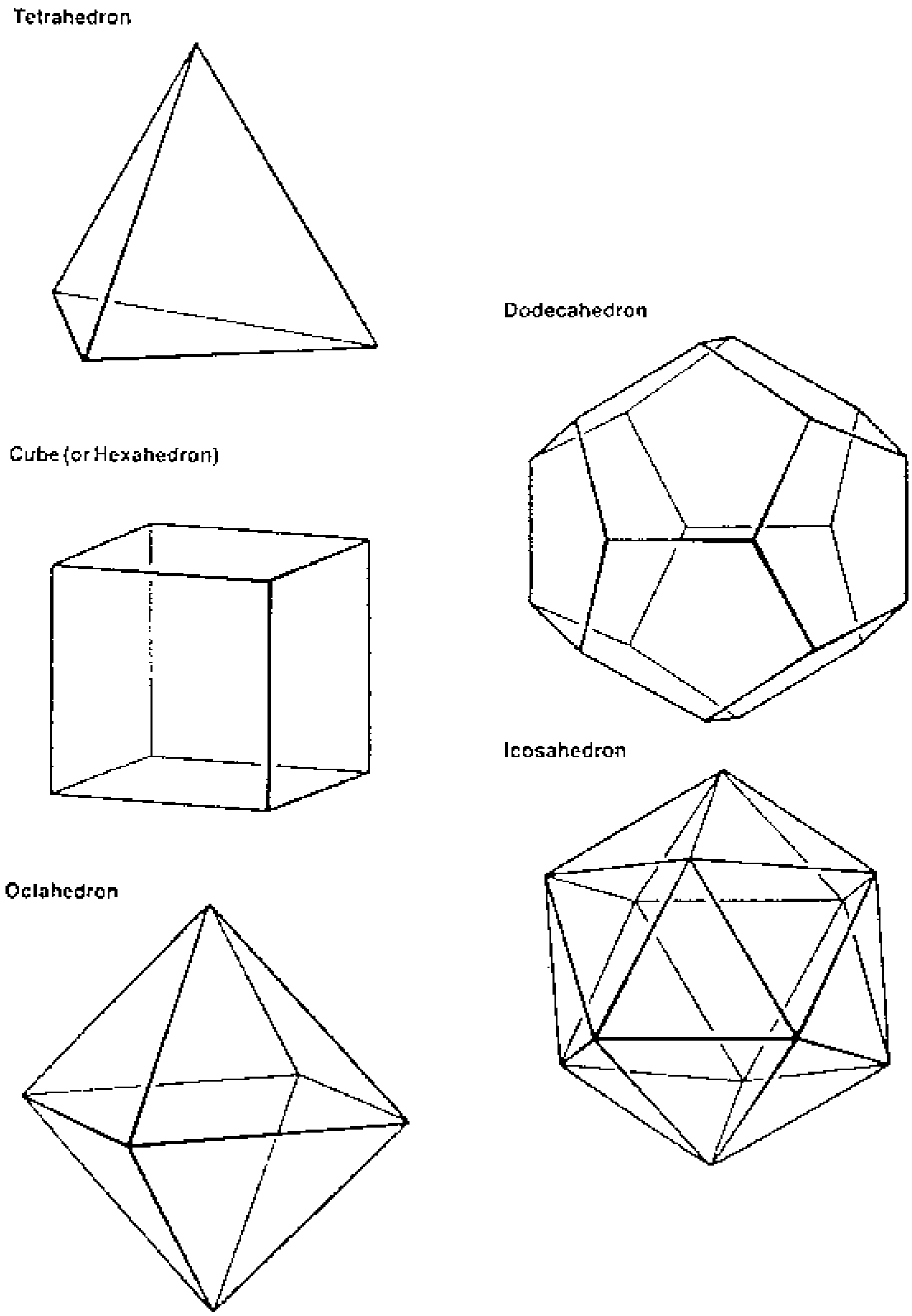

Figure 1.1: The five regular solids (platonic solids): Tetrahedron, Cube (Hexahedron), Octahedron, Dodecahedron and Icosahedron

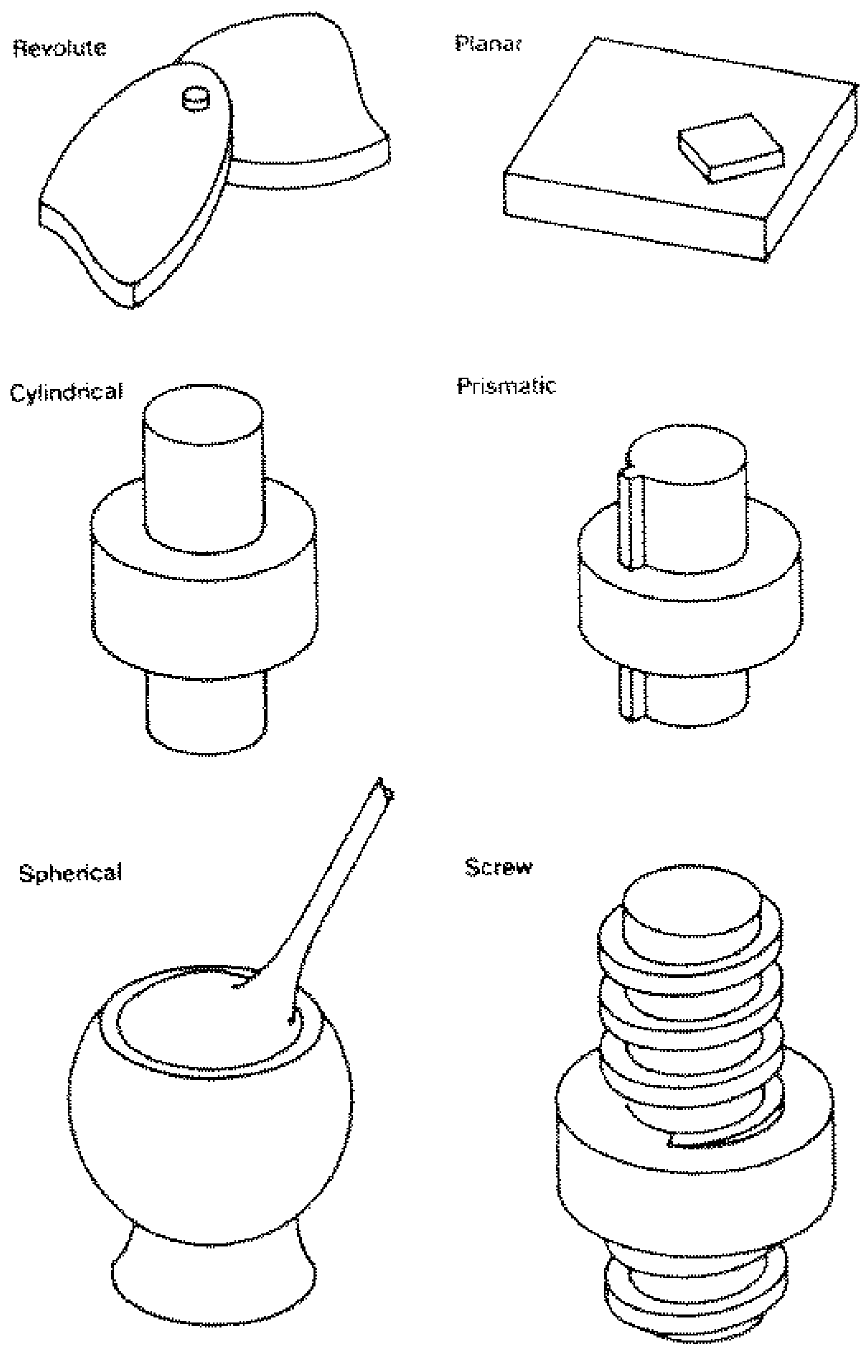

Figure 1.2: The six lower pairs

1. *g is an isometry (a distance preserving mapping), and*

2. *g leaves $S \subset \mathbf{R}^3$ setwise invariant, i.e. $g(S) = S$ where $g(S)$ denotes the set $g(S) = \{g(s) | s \in S \subset \mathbf{R}^3\}$.*

Definition 1.1.2 *A* **symmetry** *g of $S \subseteq R^3$ is a* **proper symmetry** *if g is a symmetry and g belongs to the proper Euclidean Group $\mathcal{E}^+$*

Given an arbitrary subset $S \subset R^3$, S has at least one symmetry — the identity mapping which maps each point in S to itself. Since a symmetry of an object is a transformation in space, it is important to distinguish how it acts on the whole space versus on the object to which this symmetry belongs. For a given coordinate system XYZ defined on R^3, the identity mapping, a reflection about the XZ plane and a rotation about the Z axis are three *distinct* symmetries of the Z axis. Even though they cause the same effect to the points on the Z axis (pointwise invariant), they affect the whole space of R^3 differently.

Some objects have only one possible symmetry (irregular), some have many, and yet some have an infinite number of symmetries. The set of all the symmetries of an object S forms a **group** G. The size of G is called the *order* of the group. Now let us review the mathematical notion of a *group* G:

Definition 1.1.3 *G is a set of elements that satisfy the following constraints:*

- an associative binary operation $*$ is defined on the set;
- for any pair of elements g_1, g_2 in G, $g_1 * g_2$ is also in G (G is closed);
- there exists an *identity* element e such that any element g in G, $e * g = g * e = g$;
- any element g in G has an *inverse* element g^{-1} also in G, such that $g * g^{-1} = g^{-1} * g = e$.

All the rigid transformations (translations, rotations, reflections, and combinations of them) on R^3, i.e. symmetries of R^3, form the **Euclidean Group** $\mathcal{E}$, where

- the composition of any two transformations is an **associative binary operation** over all the rigid transformations in R^3,
- the identity mapping is the **identity** transformation,
- every transformation has its reverse transformation as its **inverse**,
- any two transformations carried out one after the other remain a rigid transformation in R^3, thus the composition is **closed**.

Proposition 1.1.4 *All the symmetries of a subset $S \in R^3$ for a symmetry group G_S of S.*

Proof : ... ...

All the handedness-preserving isometries, i.e. excluding reflections in $\mathcal{E}$, form the **proper Euclidean Group** $\mathcal{E}^+$. $\mathcal{E}^+$ is a *subgroup* of $\mathcal{E}$. The symmetry group of any $S \subset R^3$ is a subgroup of $\mathcal{E}$, or of $\mathcal{E}^+$ if reflections are excluded.

Using the composition of mappings in R^3 as the binary operation $*$ defined on the set of symmetries of a $S \subseteq R^3$, one can prove that *all the symmetries of S form a group*, which is thus called the **symmetry group** of S. An object's symmetry group is one of its most important descriptors.

Proposition 1.1.5 *The proper symmetries of a set $S \subseteq \Re^3$ form a subgroup of $\mathcal{E}^+$.*

Proof :

Let G denote the set of the symmetries of $S \subset \Re^3$. Obviously, $1(S) = S$, so $1 \in G$. If $g \in G$ then $g(S) = S$, multiplying by g^{-1} we have $g^{-1}g(S) = g^{-1}(S)$ therefore $g^{-1}(S) = S$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1 g_2)(S) = g_1(g_2(S)) = g_1(S) = S$ therefore $g_1 g_2 \in G$. By the definition of a group (Definition 1.1.3) and the fact that G is a subset of $\mathcal{E}^+$, G is a subgroup of $\mathcal{E}^+$. □

Definition 1.1.6 *G_1, G_2 are subgroups of group $\mathcal{E}^+$, G_1 is* **conjugate** *to G_2 iff there exists $g \in \mathcal{E}^+$ such that $G_1 = gG_2g^{-1}$.*

Proposition 1.1.7 *If G is the symmetry group of $S \subset \Re^3$ then for any rigid transformation g in $\mathcal{E}^+$, gGg^{-1} is the symmetry group of $g(S)$.*

Proof : ... ...

Thus one way to denote the symmetry group of S computationally is to use a pair: a *canonical symmetry group* G_{canon} and an element g of $\mathcal{E}^+$ that transforms S from the origin (some standard position) to its current location. Table 1.1 shows a sample of some canonical symmetry groups in $\mathcal{E}^+$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are orthogonal unit vectors along axes X, Y, Z. $\mathbf{trans}(x, y, z)$ is a translation and $\mathbf{rot}(a, b)$ is a rotation about axis a for angle b. **gp** indicates a group instead of a set. The doubled lines divide this set of subgroups into: the identity group, rotational subgroups, translational subgroups and subgroups containing both translations and rotations. These *canonical* subgroups are chosen with respect to a specified coordinate system in a systematic way: if they have a single axis of rotation, it is chosen to be the Z-axis, if they leave a single point in R^3 fixed, that point is chosen to be the origin. If they leave a plane setwise invariant, the plane is chosen as the X-Y plane. A list of finite rotation groups are shown in Table 1.2. These are symmetry groups of the five platonic solids shown in Figure 1.1. See Figure 1.3 for the relative containment relationship among the subgroups defined in Tables 1.1 and 1.2.

Definition 1.1.8 *A* **pole** *of a rotation group $R = tR_ct^{-1}$, $R_c \subset SO(3), t \in T^3$, is a point p on the unit sphere S_0 that is left fixed by some rotation of R_c other than the identity, i.e. for some $p \in S_0, \exists r \in R_c, r \neq 1$ such that $r(p) = p$.*

Groups can be divided into different categories. For example, G is a *finite* group if there is a finite number of elements in G. Otherwise G is infinite. Infinite groups can be further divided into *discrete*, non-discrete and *continuous* groups. For example, $G_{T_1C_2}$ in Table 1.1 is not a discrete nor a continuous group but a non-discrete group. One way to define a discrete symmetry group is to use the concept of *orbit*.

Definition 1.1.9 *An* orbit *of a point $x \in R^3$ under group G, or the* G-orbit *of x, is $G(x) = \{g(x) | g \in G\}$.*

Table 1.1: Typical Members of Some Canonical Subgroups of $\mathcal{E}^+$

Canonical Groups	Definition
G_{id}	$\{1\}$
Rotation Subgroups	
$SO(3)$	$\mathbf{gp}\{\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi) \vert \theta,\sigma,\phi \in R\}$
$O(2)$	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi) \vert \theta \in R, n \in \mathcal{N}\}$
$SO(2)$	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},\theta) \vert \theta \in R\}$
D_{2n}	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},2\pi/n)\mathbf{rot}(\mathbf{i},m\pi) \vert m \in \mathcal{N}\}, n \in \mathcal{N}$
C_n	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},2\pi/n)\}, n \in \mathcal{N}$
Translation Subgroups	
$\mathcal{T}^1$	$\mathbf{gp}\{\mathbf{trans}(0,0,z) \vert z \in R\}$
$\mathcal{T}^1_{dis}(t_0)$	$\mathbf{gp}\{\mathbf{trans}(0,0,t_0)\}, t_0 \in R$
$\mathcal{T}^2$	$\mathbf{gp}\{\mathbf{trans}(x,y,0) \vert x,y \in R\}$
$\mathcal{T}^3$	$\mathbf{gp}\{\mathbf{trans}(x,y,z) \vert x,y,z \in R\}$
Mixed Subgroups	
G_{cyl}	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi) \vert n \in \mathcal{N},\theta,z \in R\}$
G_{dir_cyl}	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta) \vert z,\theta \in R\}$
G_{plane}	$\mathbf{gp}\{\mathbf{trans}(x,y,0)\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi) \vert x,y,\theta \in R, n \in \mathcal{N}\}$
G_{dir_plane}	$\mathbf{gp}\{\mathbf{trans}(x,y,0)\mathbf{rot}(\mathbf{k},\theta) \vert x,y,\theta \in R\}$
$G_{screw}(p)$	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},2z\pi/p) \vert z \in R\}, p \in R$
$G_{T_1C_2}$	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{i},n\pi) \vert n \in \mathcal{N}, z \in R\}$
$\mathcal{E}^+$	$\mathbf{gp}\{\mathbf{trans}(x,y,z)\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi) \vert x,y,z,\theta,\sigma,\phi \in \Re\}$

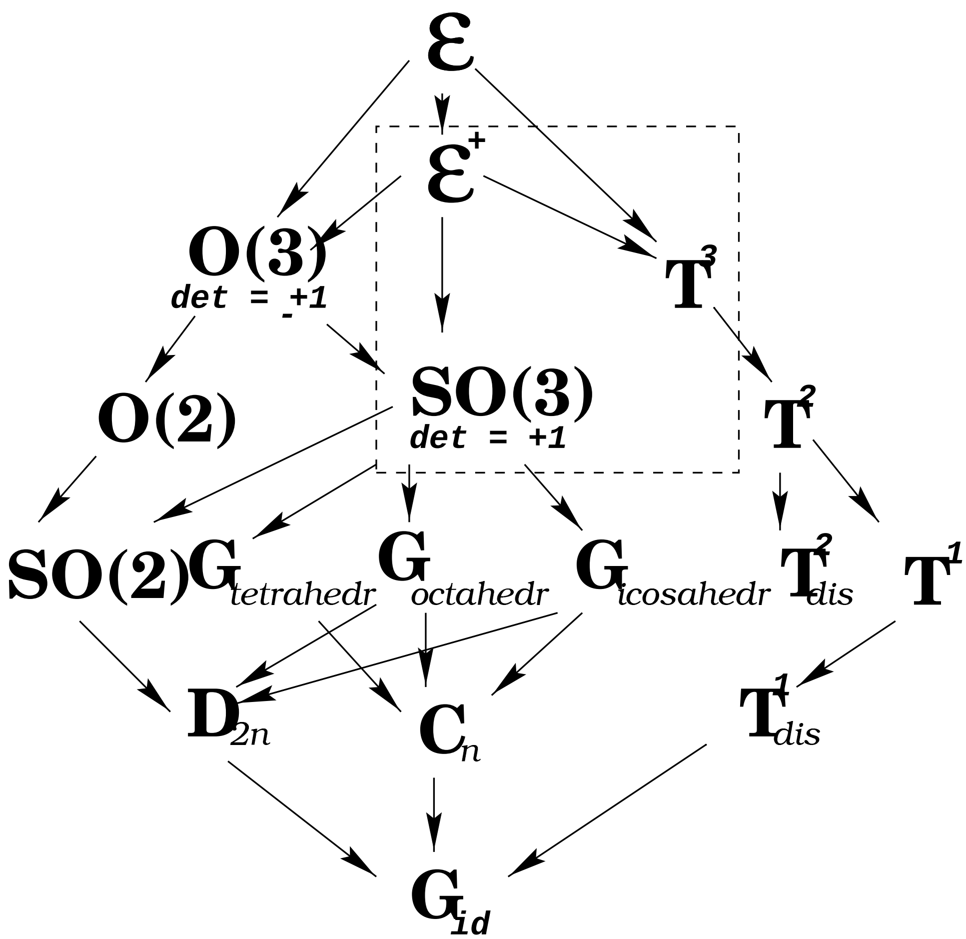

Figure 1.3: Here the arrows $A \to B$ means B is a *subgroup of* A.

Table 1.2: Types of finite rotation groups

Group Name	$\lvert G\rvert$ size	Poles in Orbit 1	Poles in Orbit 2	Poles in Orbit 3	Comments
G_{id}	1	∞	0	0	the only rotation that leaves more than two points of a solid fixed
$C_n = G_{cyclic}$	n	1	1	0	generated by the $2\pi/n$ rotation
$G_{2m} = G_{dihedral}$	$n = 2m$	$n/2$	$n/2$	2	for a regular m-gon, $m+1$ rotation axes
$G_{tetrahedral}$	12	6	4	4	7 rotation axes
$G_{octahedral}$	24	12	8	6	13 axes, same as cube
$G_{icosahedral}$	60	30	20	12	31 axes, same as dodecahedron

A discrete symmetry group can be defined in terms of its orbits:

Definition 1.1.10 *A discrete group G is a subgroup of $\mathcal{E}$ such that for any $x \in R^3$ and any sphere $B_r = \{y \mid y \in R^3, \lvert\lvert y\rvert\rvert \leq r\}$ there are only a finite number of points in the G-orbit of x that are contained in B_r.*

According to this definition, C_n is a discrete group but $SO(3)$ is not.

There are many ways to denote a group, such as, exhaustively listing all the members (e.g. the identity group for $\{1\}$) or using a multiplication table for finite groups [5], exponent representation for Lie groups [4], or generators-and-relations [10] for countable (finitely generated) groups. For the affine subgroups, for example, one can use homogeneous matrices; for the rotation subgroup only, quaternions. However, given the diversity of the groups listed in Table 1.1, to have a uniform denotation on computers for the symmetry groups of all the solids and surfaces in Euclidean space is a challenging problem. One cannot list all the members of an infinite group (e.g. the symmetry group of a sphere SO(3)), nor can generators-and-relations be used to represent a

Lie group. The exponent representation of Lie groups cannot be used to represent finitely generated groups such as crystallographic groups. Furthermore, the denotation we are seeking should also accommodate the computation of group intersections. This is the topic that will be addressed in Chapter **??**.

1.1.2 Symmetry Groups and Solids in Contact

A *solid* is a three-dimensional, connected and closed subset of Euclidean space. Each solid is bounded by a set of oriented algebraic surfaces.

Definition 1.2.11 *The* **contact region** *of two solids is composed of those points in* R^3 *that coincide with both solids.*

Two solids have a **surface contact** if their *contact region* is a 2D area. By **contacting surfaces** of a solid we mean the algebraic surfaces that coincide with the contact region and bound the solid, *not* the finite contact region itself. One important observation can be made from studying lower pairs in Figure 1.2: *When two solids have a surface contact, the contacting surfaces from the different solids must have the* **same** *symmetry group.* In other words: Having the same symmetry group for the contacting surfaces of two solids is a necessary condition for any two solids to have a *surface contact.*

There is a direct relationship between the symmetries of the contacting surfaces and the possible relative motions between a pair of solids under such a contact, as can be observed in Figure 1.2. For a simpler and incremental case see Figure 1.4, where block A has one surface in contact with its environment, block B has two, and block C has three. One can uniquely and completely describe three different kinds of relative motions between the block and its environment under these three surface contacts. Block A can be translated horizontally as well as rotated about any axis that is perpendicular to the contacting planar surface — the symmetry group of the plane parallel with the contacting surfaces of the two solids. Similarly, block B can be translated along the baseline where the wall and the floor intersect — the symmetry group of the pair of planes that coincide with the wall and the floor. Block C's position is fixed when three surfaces of the block sharing one vertex are chosen.

Given the symmetries of each block (equivalence of its surfaces), there are multiple choices on block's contacting surfaces with its environment. If the blocks are cubes, A has six choices and B has 24 (for each of the six surfaces there are four ways to choose another surface which shares an edge with the chosen one, or there are six edges on the cube and each has two possible orientations.). Figure 1.5 demonstrates the contact of block C as a cube. When instructing a robot to put a cube in a corner there are 24 different possible positions for the cube to be in. This is obtained by finding out that for each of the eight vertices there are three choices for contacting surfaces, $8 \times 3 = 24$. 24 is exactly the size of the symmetry group of a cube. That is why the simple task of putting a cube in a corner requires 24 different sets of task specifications to be unambiguous and complete.

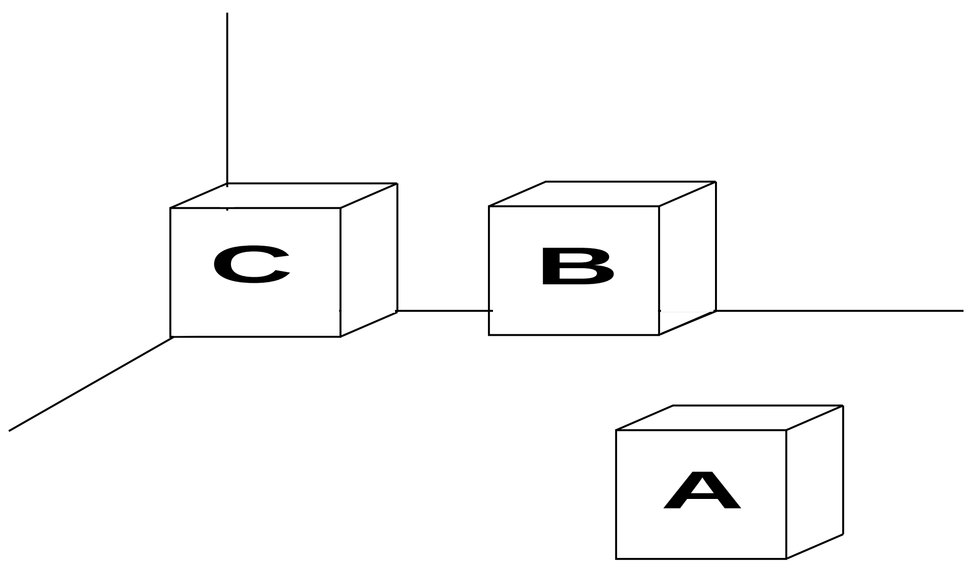

Figure 1.4: Block A has one contacting surface pair, block B has two and block C has three

When one defines a coordinate system for each solid with respect to the world coordinate system, one can talk about the *relative position* of two solids, which is the transformation from the coordinate system of one solid to that of the other. The relative positions of two solids under any surface contact can be derived via the coordinates of the contact-

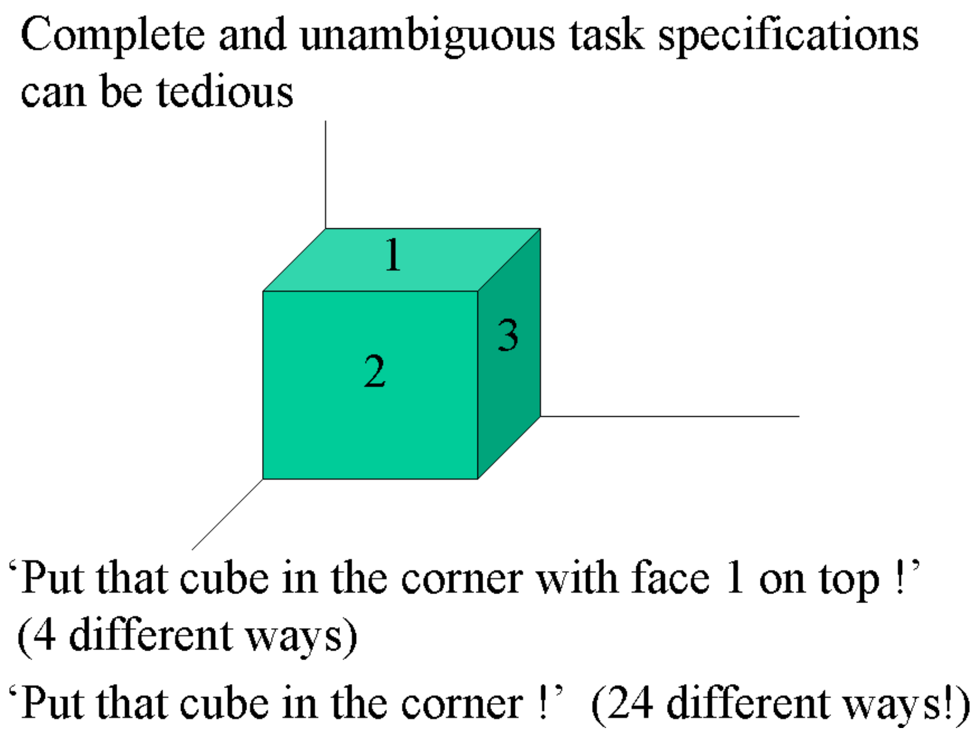

Figure 1.5: "Put a cube in a corner" is an ambiguous task specification, and can be tedious when completeness is desired.

ing surfaces and their symmetry groups. Figure 1.6 shows the relative location of a surface feature with respect to the solid it resides on when surface coordinates are defined. The **relative position** of solid A

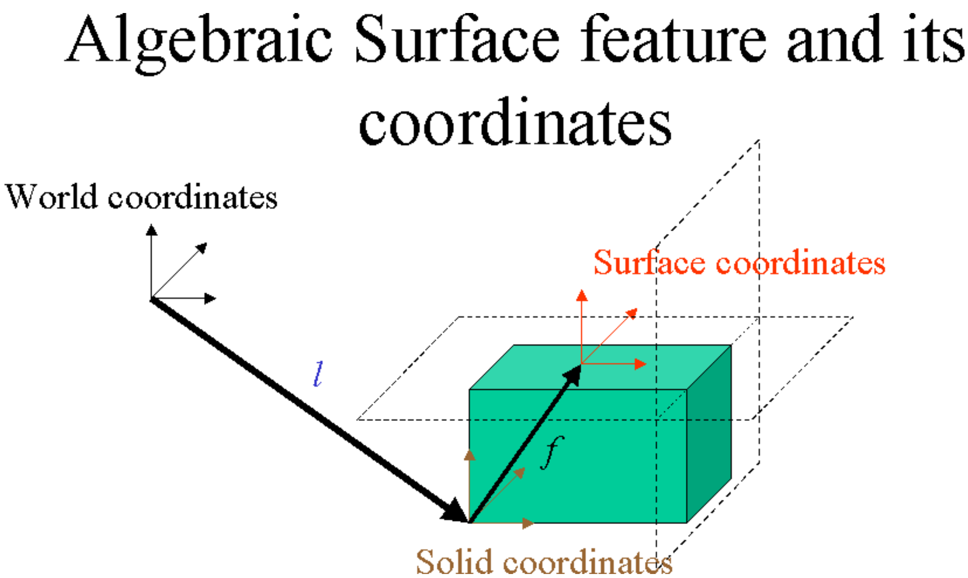

Figure 1.6: A set of surface coordinates are defined with respect to its solid coordinates. The solid coordinates are defined with respect to those of the world.

with respect to solid B is a transformation from the coordinate system of solid B to that of solid A. The **relative position** of solid A with respect to solid B **under a contact** consists of all possible relative positions of A with respect to B that maintain their contact region. If solid A is acted upon by the symmetries of its contacting surfaces the *same* contact *remains*. This can be seen by a simple example of a sphere sitting on a table (Figure 1.7). Any rotations about the center of the sphere (symmetries of the sphere) do not break the contact between the sphere and the table. Similarly, when any symmetries of the planar surface of the table act on the table, the contact or the spatial relationship between the table and the sphere remains. Contrast this surface to surface contact with the following cases:

- putting the apex of a cone on a planar surface requires the singular point of the conic surface to be in contact with the plane, thus

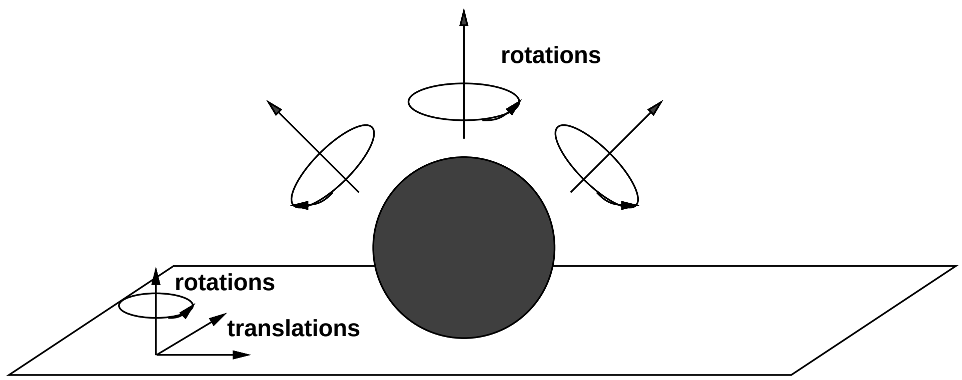

Figure 1.7: The contact between a spherical surface and a planar surface remains the same when the symmetries of the contacting surfaces are applied to each solid.

the symmetries that keep the contact are those of the apex of the cone, not just those of the cone itself;

- putting one edge of a block on a table requires the intersection of two planar surfaces of the block to be in contact with the planar surface of the table, so only the symmetries of this intersection — a line — can act on the block and keep the contact.

The following definitions are stated for surface to surface contact but can be extended to non-surface contacts as described above.

Let B_1 and B_2 be two solids, with surfaces F_1 and F_2 in contact. G_1, G_2 are the symmetry groups of S_1, S_2 respectively. Suppose l_1, l_2 specify the locations of solids B_1, B_2 in the world coordinate system and f_1 and f_2 specify the locations of F_1, F_2 in their respective body coordinates (Figure 1.8). A *spatial relationship* between two solids in contact is a binary relation $\tau \subset \mathcal{E}^+ \times \mathcal{E}^+$, where each pair $(l_1, l_2) \in \tau$ specifies one pair of possible positions for B_1 and B_2. Regardless of what kind of contacts occur between F_1 and F_2, if we move B_1 or B_2 by a member of G_1 or G_2, respectively, the relationship between the contacting surfaces remains[2]. Consider what we can infer about the

[2]The motion may not be legal, however, since it may cause inter-penetration of solids.

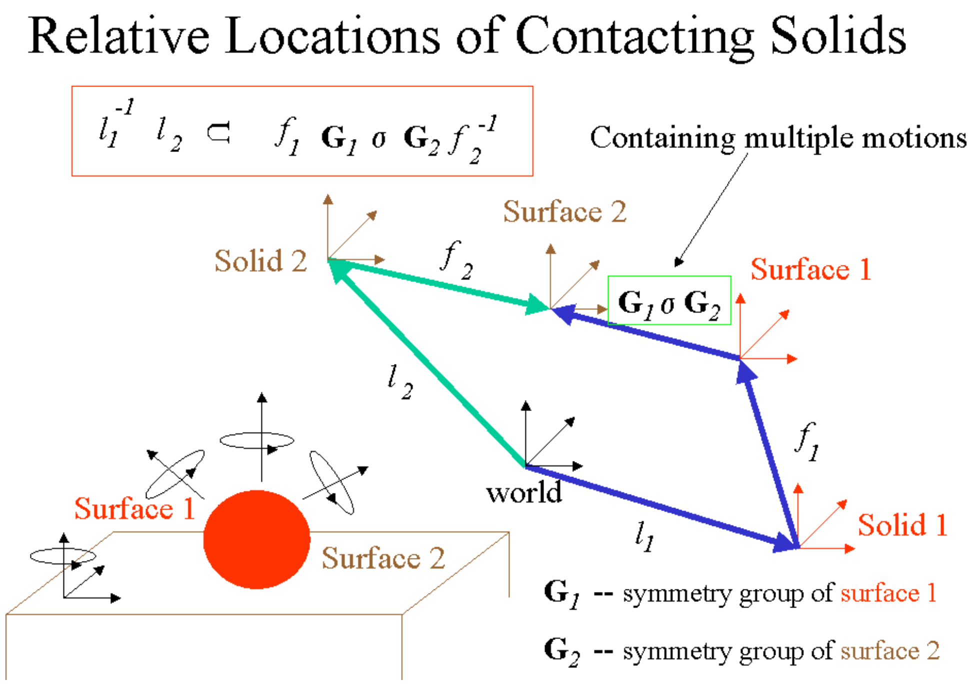

Figure 1.8: The relative locations of two solids *solid*1, *solid*2, in contact through their surfaces F_1, F_2, are expressed in terms of their coordinates and their symmetry groups G_1, G_2 respectively: $l_1^{-1} l_2 \in f_1 G_1 \sigma G_2 f_2^{-1}$.

relative location of two solids that are in contact. The relative location of B_2 with respect to B_1, i.e. $l_1^{-1}l_2$, can be expressed as[3] (Figure 1.8):

$$l_1^{-1}l_2 \in f_1 G_1 \sigma G_2 f_2^{-1}, \tag{1.1}$$

where σ is a rigid transformation which brings the coordinate of F_1 into coincidence with the coordinate of F_2. Since one can define a coordinate for a surface arbitrarily, σ can always be made to be the identity element. However, in practice the coordinate is usually defined in a convenient way (centrally located, for instance).

If the two solids have a *surface contact* through surfaces F_1, F_2, the contacting surfaces coincide. F_1, F_2 are then the same subset in R^3, and thus have the same symmetry group. In this case expression (1.1) reduces to a much simpler form:

$$l_1^{-1}l_2 \in f_1 G f_2^{-1}; \tag{1.2}$$

where σ is now the identity and $G = G_1 = G_2$.

The specific surface contact can be expressed as a spatial relationship τ between solids B_1, B_2

$$\tau = \{(l_1, l_2) | l_1^{-1}l_2 \in f_1 G f_2^{-1}\}. \tag{1.3}$$

We can summarize this by saying that *when two solids have a surface contact, their relative locations belong to a two-sided coset of the common symmetry group of the contacting surfaces.*

If G is of infinite order, say $G = SO(2)$, then τ can take infinitely many values; this is the case for a revolute joint (Figure 1.2). If G is finite, then there is a finite number of relative locations possible between the two solids. If G is the identity group, then τ has only the constant value $f_1 f_2^{-1}$. In this case the location of one solid with respect to the other is uniquely determined. Under this formulation, we have the simplest form for the most asymmetrical case.

Two solids in an assembly are typically related to each other through multiple surface contacts. If two solids B_1 and B_2 are related by two pairs of surface contacts, i.e. F_{11} fits F_{21} and F_{12} fits F_{22} with surface

[3] If one uses 4x4 homogeneous matrices to express the group actions in formula 1.1, the order should be reversed.

locations $f_{11}, f_{21}, f_{12}, f_{22}$, in their body coordinate systems, then the relative location of body B_2 to body B_1 is constrained by two relations of the form (1.2), that is:

$$l_1^{-1} l_2 \in f_{11} G_{11} f_{21}^{-1} \bigcap f_{12} G_{12} f_{22}^{-1}; \tag{1.4}$$

where G_{11} is the symmetry group of F_{11} (F_{21}), G_{12} is the symmetry group of F_{12} (F_{22}), and σ is the identity. Expression (1.4) can be rewritten into a one-sided coset of an intersection involving the original groups G_{11} and G_{12} [27]. When the intersection is null, it implies that the specified spatial relationship is impossible (i.e. no feasible positions l_1, l_2 can realize the required contacts between B_1 and B_2). If the spatial relationship is realizable, the two surfaces of B_1 can be viewed as one *compound feature* F_1 composed of F_{11} and F_{12} fitting with another compound feature F_2 composed of F_{21} and F_{22} of B_2. The symmetry groups G_1 of F_1 and G_2 of F_2 should be the same since F_1, F_2 coincide, let us call this common symmetry group G. Following (1.2), we have:

$$l_1^{-1} l_2 \in f_1 G f_2^{-1}, \tag{1.5}$$

where f_1, f_2 are the locations of compound features F_1, F_2 with respect to their body coordinate systems. The questions now are: what is G? and how do we find G from G_{11}, and G_{12}?

The more general questions are:

1. Given two solids S_1, S_2, what is the relative location of the two under n surface contact (n contacts from each side, Figure 1.9)?

2. Given two solids S_1, S_2, what is the relative location of the two under n general contact?

3. Given m solids in a chaining general contact (Figure 1.10), what is the relative location of the mth solid with respect to the first solid?

From our reasoning so far we can express the relative locations in each case above using the symmetry group of the contacting surfaces:

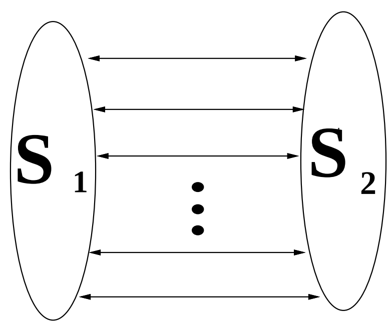

Figure 1.9: Solids S_1 and S_2 have n contacts

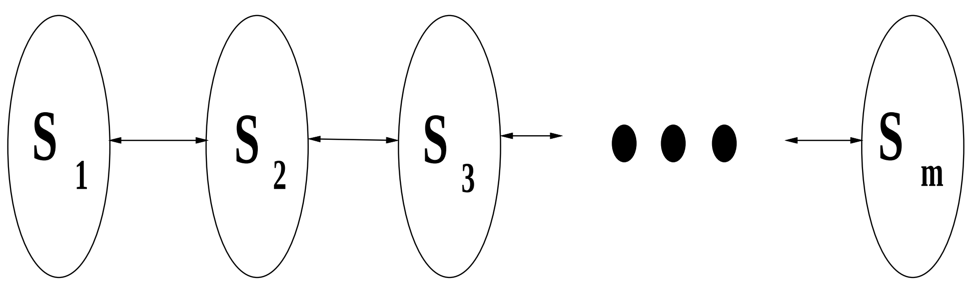

Figure 1.10: Solids $S_1, S_2, ...S_m$ form a chain

1. Two solids have n surface contact:

$$l_1^{-1}l_2 \in f_1 G f_2^{-1} \tag{1.6}$$

   where G is the symmetry group of *all* the contact surfaces of S_1 or S_2.

2. Two solids have n general contact:

$$l_1^{-1}l_2 \in f_{11}G_{11}\sigma_1 G_{21} f_{21}^{-1} \cap f_{12}G_{12}\sigma_2 G_{22} f_{22}^{-1} \cap \ldots \tag{1.7}$$

$$\cap f_{1n}G_{1n}\sigma_n G_{2n} f_{2n}^{-1}$$

   where G_{ij} is the symmetry group of primitive feature j of S_i and f_{ij} is its feature coordinates.

3. m solids have a chaining general contact, the relative location of solid m with respect to solid 1:

$$l_1^{-1}l_m \in f_1 G_{12}\sigma_1 G_{21} f_{21}^{-1} f_2 G_{23}\sigma_2 G_{32} f_{32}^{-1} \ldots \tag{1.8}$$

$$f_{m-1}G_{(m-1)m}\sigma_{m-1}G_{m(m-1)}f_{m(m-1)}^{-1}$$

   where G_{ij} is the symmetry group of the surface on solid i in contact with solid j.

1.2 Descriptives for Formalization

Understanding that group theory is relevant in describing solids in contact is only the first step towards finding a solution to the problem of dealing with local and global symmetries. The next step is to establish the basic vocabulary to formalize the problem. Since solids are in contact with each other via the finite faces bounding their volumes, these faces seem to be able to serve as the primitives. However, these bounded faces are only a physical realization of the contact between the sometimes unbounded algebraic surfaces that coincide with the finite faces. When a surface contact is specified between only one pair of planar primitive features of two solids, no real physical contact of the two solids is enforced (Figure 1.11). To guarantee a physical contact, additional constraints have to be added. This is precisely what is meant by a pair of planar surface contacts. Assuming g is a symmetry of some volume $S \subseteq R^3$ (S can be unbounded) and $\bar{S}$ is the complement of S, i.e. $S = R^3 - \bar{S}$, then $g(S) = S = R^3 - \bar{S}, g(\bar{S}) = g(R^3 - S) = g(R^3) - g(S) = R^3 - S = \bar{S}$. That is to say, S and its complement $\bar{S}$ have the same symmetry group — the symmetries of the common boundary of S and its complement. Thus, when describing contacts and motions happening along the boundary of a finite solid in R^3, it is reasonable and more precise to formalize it as events happening to a set of possibly infinite surfaces bounding the solid.

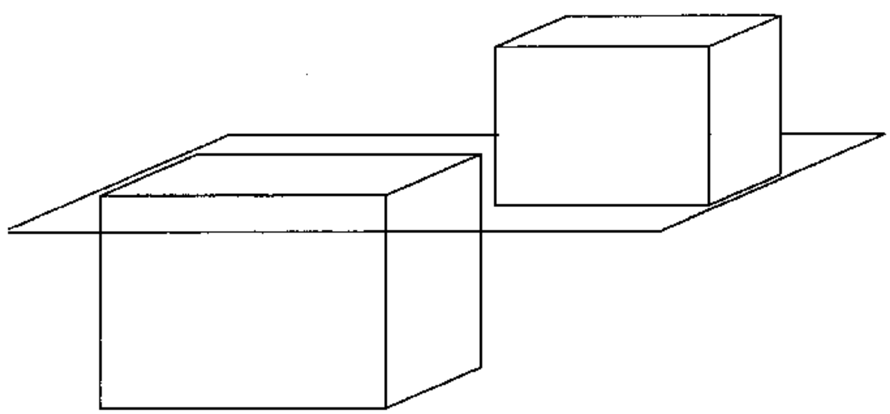

Figure 1.11: Two solids which share a planar surface feature

1.2.1 Oriented Surface Primitives

Since contacts among solids happen via the contacts of the surfaces of the solids, the representation and characterization of each contacting surface constitutes the foundation of any formalization for solid contacts. A group theoretic formalization of surface contact [31] treats each surface of a solid as a (possibly infinite) subset S in Euclidean space, which can be expressed as a polynomial. A symmetry of S is defined as follows:

A single surface, treated as a set of points or with orientation vectors pointing inward, has the same symmetries as the surface with orientation vectors pointing outward[4]. However, in real world problems it is rare that only one surface is considered in isolation. In an assembly, it is often the case that multiple surfaces of one solid are in contact with multiple surfaces of other solids. This is a situation where treating a surface as a set can run into problems. For example, Figure 1.12 shows

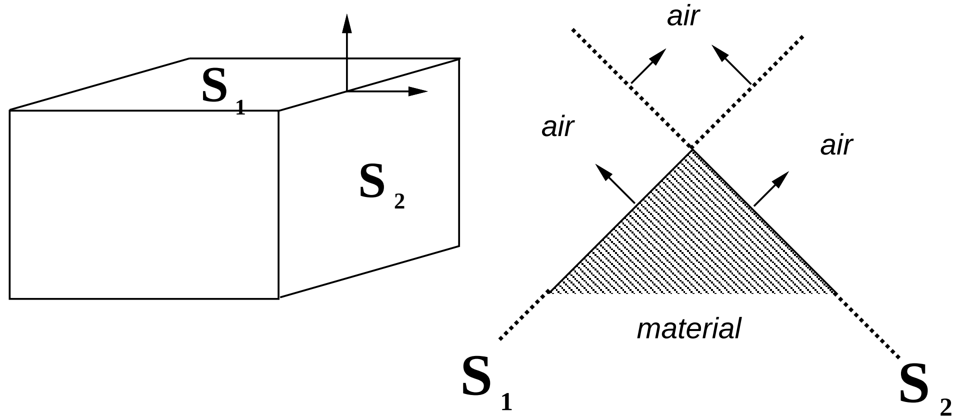

Figure 1.12: Two adjacent planes, S_1, S_2, on a cube

two adjacent (infinite) planar surfaces S_1, S_2 of a block. If the two surfaces are treated as sets, the symmetries of the two planes include a 90^0 rotation about the line of the intersection of the two planes, which is not a symmetry in reality. If one takes into consideration the fact that one side of the plane is the material of the solid and the other is the air, the only symmetries left are those 180^0 rotations that preserve both the bounding surfaces and their orientations. Another example of such non-real symmetries is illustrated in Figure 1.13. If the two cylindrical surfaces S_1, S_2 are treated as sets (infinite cylinders) then one cannot distinguish the two cases (a) and (b). In case (b) the cylindrical hole S_1 and the cylinder S_2, though they have the same radius, are not interchangeable if one takes their orientations into consideration.

[4]Planar surfaces are an exception: when an unoriented plane is treated as a set, there are flipping symmetries that do not exist for an oriented plane. In practice, this can be easily handled by checking the signs of a surface given by a solid modeler.

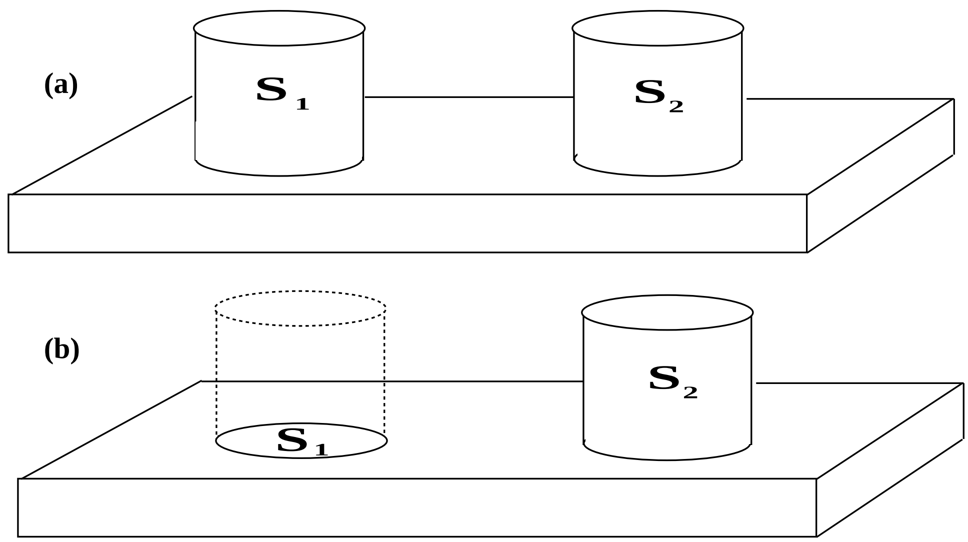

Figure 1.13: In both cases (a) and (b), S_1 and S_2 are interchangable (2-congruent) if they are treated as sets.

Obtaining the accurate symmetry group of a set of oriented contacting surfaces becomes crucial in applications where an assembly planner needs to decide which assembly parts fit with each other [32, 27] based on whether they have compatible symmetry groups. It is also crucial when instructing a robot since the exact relative positions/motions between a subassembly and the rest of the assembly have to be provided in the task specification. These applications call for a more precise characterization of surface features of a solid, i.e. taking the orientations of a surface into consideration. This addition to a set-feature will require that the symmetries of a surface keep both the points on the surface and the orientations of the surface, respectively, setwise invariant. The group theoretical formalization, thus, needs to be re-evaluated given oriented surfaces as the descriptive primitives of a solid.

We introduce the concept of *oriented features* by defining a set of outward-pointing normal vectors for each surface point of a solid. The polynomial used to express an algebraic surface implicitly, such as provided by a geometric solid modeler, defines such normal vectors. Let $\mathcal{S}^2$ be the unit sphere at the origin embedded in $\Re^3$, each point of $\mathcal{S}^2$

corresponds to a unit vector in $\Re^3$.

Definition 2.1.12 *A solid M is a connected, rigid, three dimensional subset of Euclidean space $\Re^3$.*

Definition 2.1.13 *An* **oriented primitive feature** $F = (S, \rho)$ *of a solid M is an oriented surface where*

1) $S \subset \Re^3$ is a connected, irreducible[5] *and continuous algebraic surface which partially or completely coincides with one or more finite oriented faces of M;*

2) $\rho \subset S \times \mathcal{S}^2$ is a continuous relation. For each $s \in S$ if s is a non-singular point of surface S (p.78 [13]) then $v \in \mathcal{S}^2$ is one of the two opposing normals of the tangent plane at point s such that $(s, v) \in \rho$; if s is a singular point of S (e.g. at the apex of a cone) then, for all v, where $v \in \mathcal{S}^2$ is the limit of the orientations of its neighborhood, $(s, v) \in \rho$.

3) For all $s \in M, (s, v) \in \rho, v$ points away from M.

Intuitively speaking, a feature is composed of both "skin", S, and "hair", the set of normal vectors which correspond to the points on $\mathcal{S}^2$. Each element of relation ρ is a correspondence between a point on S and a unit vector on $\mathcal{S}^2$. Note, there may be more than one 'normal vector' at one point of a surface, e.g. at the apex of a conic surface.

Let $\mathcal{E}^+$ be the proper Euclidean group which contains all the rotations and translations in $\Re^3$, $\mathbf{T}^3$ be the maximum translation subgroup of $\mathcal{E}^+$, and $SO(3)$ all the rotations about the origin. We now define how an isometry $g \in \mathcal{E}^+$ acts on the relation ρ defined in Definition 2.1.13:

Definition 2.1.14 *Any isometry $g = tr$ of $\mathcal{E}^+, t \in \mathbf{T}^3, r \in SO(3)$ acts on ρ in such a way that $(s, v) \in \rho \Leftrightarrow (gs, rv) \in g * \rho$.*

Now we define the symmetries for an oriented surface:

[5]Here *irreducible* implies that a primitive feature cannot be composed of any other *more basic* algebraic surfaces.

Definition 2.1.15 *A proper isometry $g \in \mathcal{E}^+$ is a* **proper symmetry of an oriented surface** *(primitive feature) $F = (S, \rho)$ if and only if $g(S) = S$* **and** *$g * \rho = \rho$.*

Note, the difference between the symmetries of a set (Definition 1.1.1) and this definition. There is an extra demand on a symmetry for an oriented surface — it has to preserve the orientations of the surface as well. Since orientations are points on $\mathcal{S}^2$, symmetries of an oriented feature have to keep **two** sets of points in $\Re^3$ setwise invariant.

Lemma 2.1.16 *For all $g_1, g_2, g_3 \in \mathcal{E}^+$, $(g_1 g_2) * (g_3 * \rho) = g_1 * ((g_2 g_3) * \rho)$.*

Proof:

Given the associativity and the closeness property of $\mathcal{E}^+$ we have $(g_1 g_2) * (g_3 * \rho) = (g_1 g_2 g_3) * \rho = g_1 * ((g_2 g_3) * \rho)$. □

In proposition 1.1.5 it has been proven that The proper symmetries of a set $S \subseteq \Re^3$ form a subgroup of $\mathcal{E}^+$. One can prove that the symmetries of an oriented surface form a group:

Proposition 2.1.17 *The symmetries of an oriented feature $F = (S, \rho)$ form a subgroup of $\mathcal{E}^+$, called the* **symmetry group of feature** *F.*

Proof :

Let G denote the set of the symmetries of F. Since it has been shown in Proposition 1.1.5 that it is true for set S, here we only concern ourselves with ρ.

Obviously, $1 * \rho = \rho$, so $1 \in G$. If $g \in G$ then $(g * \rho) = \rho$ (By the definition of symmetries). Multiplying by g^{-1} we have $g^{-1} * (g * \rho) = g^{-1} * \rho$. Using Lemma 2.1.16 we have $g^{-1} * \rho = \rho$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1 g_2) * \rho = g_1 * (g_2 * \rho) = g_1 * \rho = \rho$ therefore $g_1 g_2 \in G$. Hence G is a subgroup of $\mathcal{E}^+$. □

An assembly is a manifestation of surface interactions of its subparts, albeit the physical property of each individual part (rigid or deformable) or the nature of the contact (static or articulated). Thus the representation of an assembly reduces to specifying a set of *contact constraints* that dictate the configuration of a set of solids. It is often the case in robotics and mechanical design that several surfaces of a solid are in contact with one or more other solids. The possible motions of this solid under these contacts are determined by the symmetry group of the contacting surfaces considered collectively. We need a denotation for such a collection:

Definition 2.1.18 *A* **compound feature** $F = (S, \rho)$ *of primitive features* $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$, *is defined to be*

- $S = S_1 \cup ... \cup S_n$
- $\rho = \rho_1 \cup ... \cup \rho_n$

The advantage of using a relation ρ to denote the orientations of a feature (Definition 2.1.13) becomes more obvious for compound features. When two primitive features are combined, there often are multiple normal directions at the points where the surfaces meet (Figure 1.14).

1.2.2 Pairwise Relationship of Oriented Surfaces

In order to determine the symmetry group of a compound feature systematically, we start with the simplest compound feature — a compound feature composed of only one pair of primitive features. See Figures 1.14, 1.15 and 1.16 for examples of simple compound features (Note that only a finite face on each primitive feature is drawn).

Given a pair of primitive features, what kind of relationship holds between the two features and what is the effect of such a relationship in terms of determining their symmetry group? The following definition gives a characterization of four relationships between a pair of primitive features:

Definition 2.2.19 *Two oriented primitive features* $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ *are said to be*

- **Distinct:** if for any open subsets $S_1' \subset S_1, S_2' \subset S_2$, no $g = tr \in \mathcal{E}^+$ exists such that $g(S_1') \subset S_2$ or $g(S_2') \subset S_1$. See Figure 1.14 for an example of a pair of distinct features F_1, F_2.

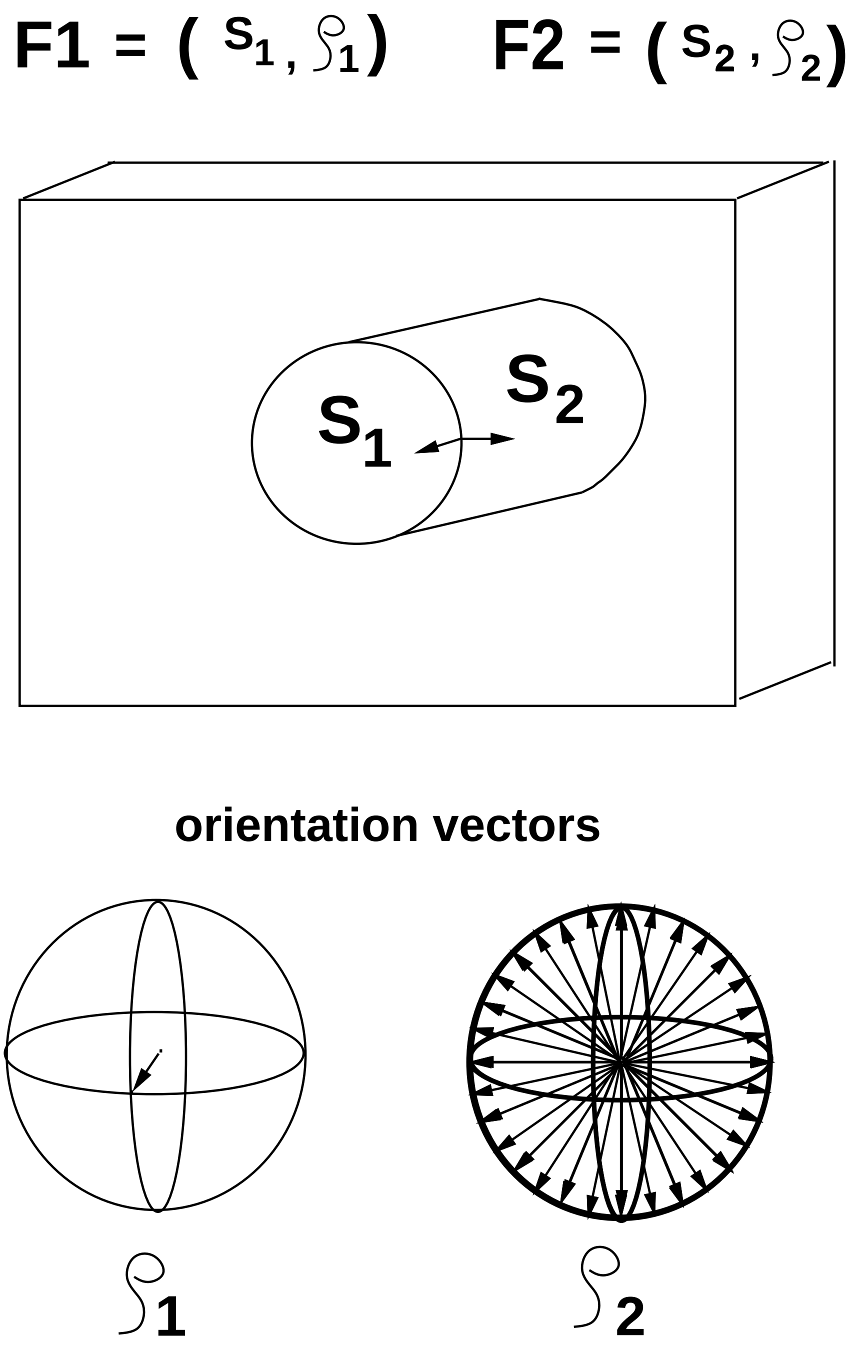

Figure 1.14: A pair of **distinct** features F_1, F_2

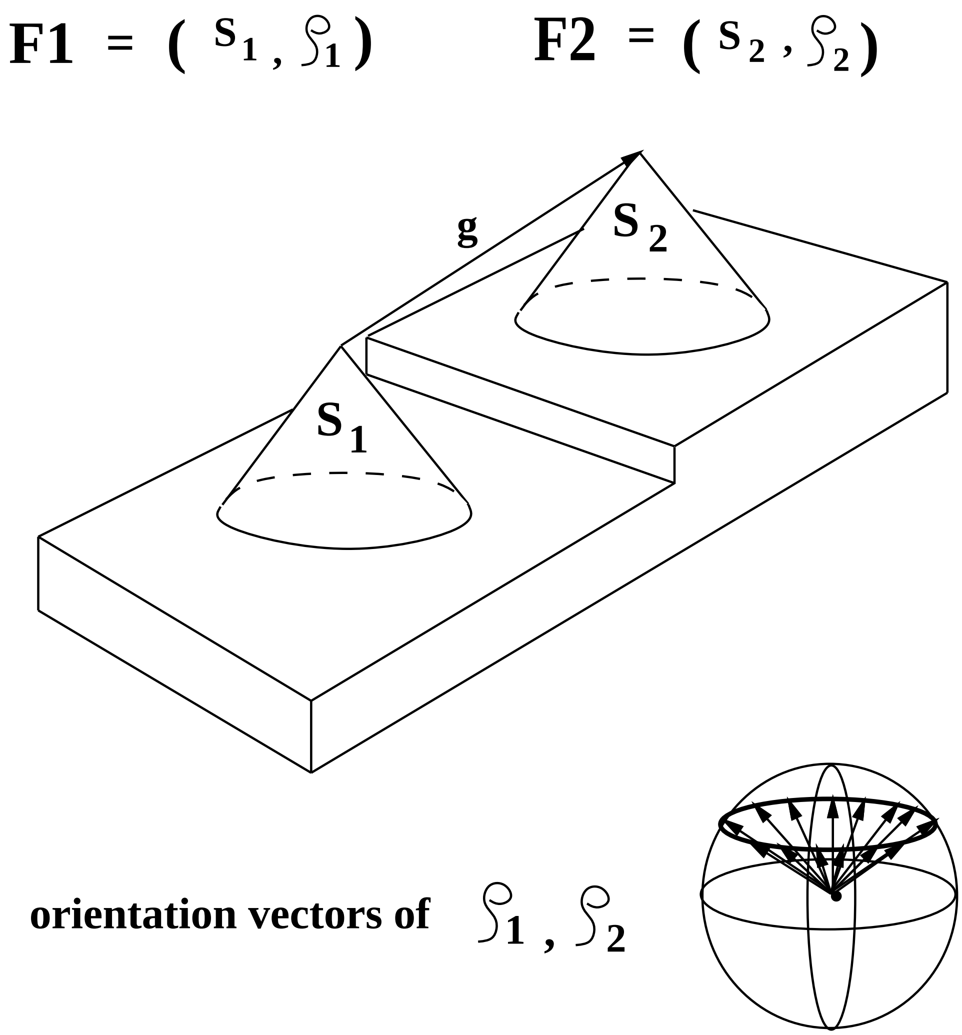

Figure 1.15: Two conic features F_1, F_2 which are **1-congruent** to each other

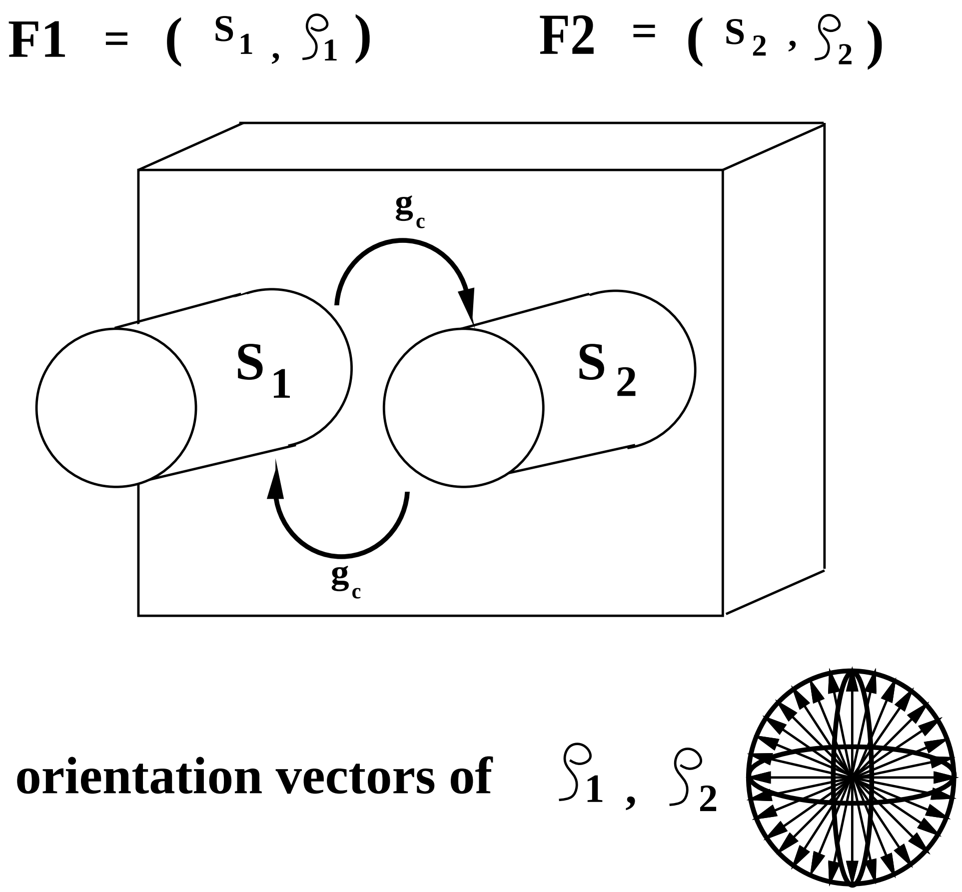

Figure 1.16: Two cylindrical features F_1, F_2 which are **2-congruent** to each other

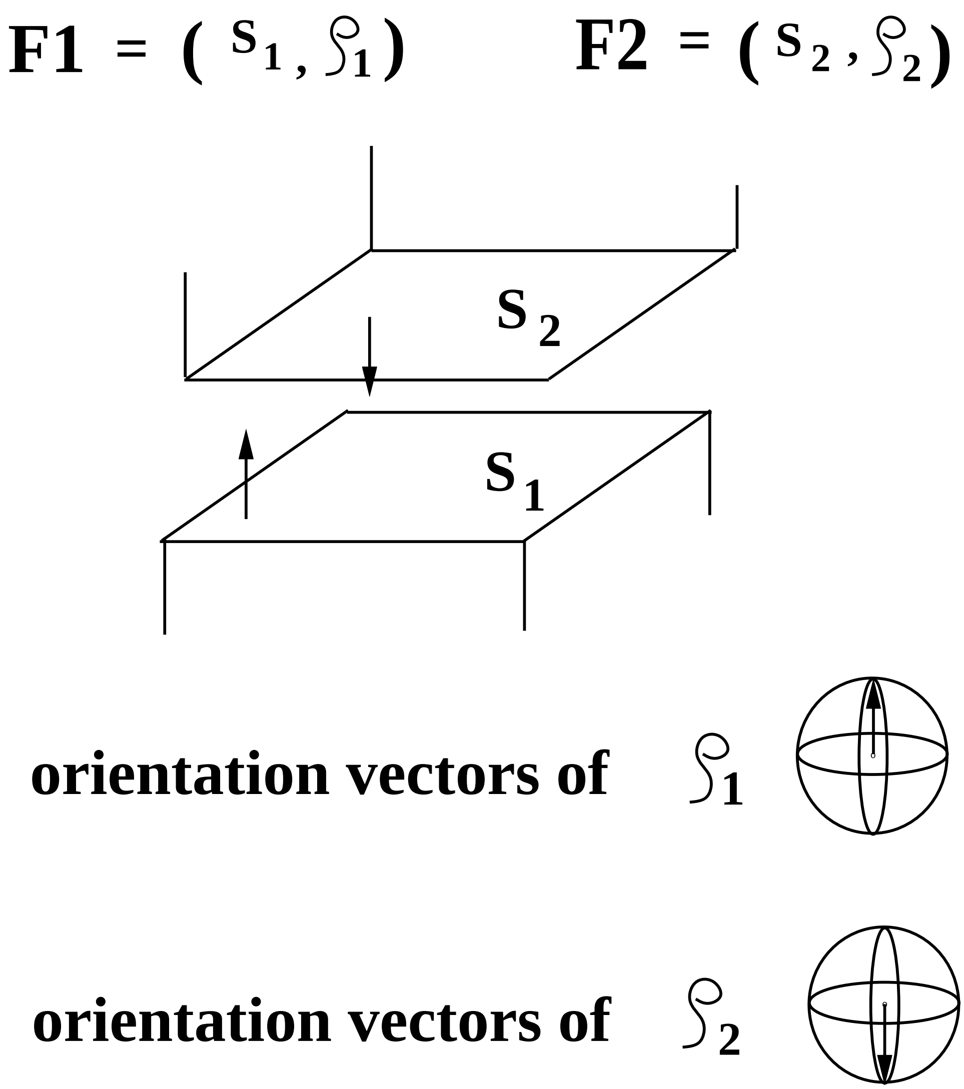

Figure 1.17: Two complementary features F_1, F_2

- **1-congruent**: if there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = \rho_2$, but for all such $g, g(S_2) \neq S_1$. For an example see Figure 1.15. Another example is two parallel planar surfaces with normal vectors pointing in the same direction.

- **2-congruent**: if there exists $g_c \in \mathcal{E}^+$ such that $g_c(S_1) = S_2, g_c(S_2) = S_1, g_c * \rho_1 = \rho_2$ and $g_c * \rho_2 = \rho_1$. For an example, consider two parallel cylindrical surfaces having the same radius and normal vectors pointing away from their center lines, as in Figure 1.16. Also, two parallel planar surfaces with normal vectors pointing to the opposite directions serve as examples of a pair of 2-congruent features. Note, such g_cs are not necessarily unique.

- **Complementary:** if there exists $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = -\rho_2$ where $-\rho_2 = \{(s, -v)|(s, v) \in \rho_2\}$; in other words, $\forall(s, v) \in g * \rho_1, \exists(s, -v) \in \rho_2$, and $\forall(s, v) \in \rho_2, \exists(s, -v) \in g * \rho_1$. See Figure 1.17 for an example.

It is easy to verify that these relationships are symmetrical and exhaustive.

The definition for oriented features allows us to distinguish a feature from its complement, which we cannot do for features treated only as sets. In general, the relationship between two primitive features can be either distinct, 1-congruent, 2-congruent or complementary. The exception is a pair of planar surfaces, which are always complementary to each other and at the same time can be either 1-congruent or 2-congruent. When two solids have a surface contact, it is the case that two features that are complementary to each other are brought into coincidence. The following proposition states how the symmetry groups of a pair of complementary features are related to each other.

Lemma 2.2.20 *Given two primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$. If there exists an open set O such that $O \subset S_1 \cap S_2$ then $S_1 = S_2$. In another words, if S_1, S_2 are locally identical then they are identical globally.*

Proof :

Analytic functions have the property that if they are locally identical then they are globally identical [1]. In the definition of primitive features (Definition 2.1.13), S_1, S_2 are defined by irreducible algebraic functions, which form a subset of the analytic functions, and thus they inherit the property. Therefore if S_1 and S_2 share an open set then $S_1 = S_2$. □

Lemma 2.2.21 *Given two primitive features* $F_1 = (S_1, \rho_1)$*, and* $F_2 = (S_2, \rho_2)$*. If there exists* $g \in \mathcal{E}^+$ *such that* $g(S_1) = S_2$ *then either* $g * \rho_1 = \rho_2$ *or* $g * \rho_1 = -\rho_2$*.*

Proof :

By Definition 2.1.13, any point s on a primitive feature has either

- a non-singular point with a unique tangent plane:

  there are two possible antipodal normals for each plane, say $v, -v$. By the definition of a primitive feature either $(s, v) = (s_2, v_2) \in \rho_2$ or $(s, -v) = (s_2, v_2) \in \rho_2$. Since ρ_1, ρ_2 are continuous mappings and isometry g does not change their continuity, for $(s_1, v_1) \in \rho_1$,

  - if $rv_1 = v_2$ then $g * \rho_1 = \rho_2$,
  - if $rv_1 = -v_2$ then $g * \rho_1 = -\rho_2$; or

- a singular point with an infinite number of "tangent planes":

  there is an infinite set of normals which are determined by the neighborhoods of the singular point s. Each of such neighborhoods is composed of non-singular points, Thus the above argument also applies.

□

Proposition 2.2.22 Distinct, 1-congruent, 2-congruent *and* complementary *are the only possible relationships between a pair of primitive features.*

Proof :

Given two primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$. Lemma 2.2.20 suggests that either there exists a $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ or no such g exists. Now let us check each case.

Note that any two planar surfaces are complementary of each other, and are either 2-congruent (when the planes intersect or are parallel with their normals pointing in the opposite directions), or 1-congruent (when the planes are parallel with their normals pointing in the same direction). In the following discussion we exclude the case of a pair of planar surfaces.

- If there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$:
  - If $g(S_2) = S_1$ also, then $g(S_1) = g(g(S_2)) = S_2 \Rightarrow g^2 = 1$. Now there are two cases in terms of their orientations (Lemma 2.2.21):

    If $g * \rho_1 = \rho_2$ then $g * \rho_2 = g * g * \rho_1 = \rho_1$. This is the definition of **2-congruent**.

    If $g * \rho_1 = -\rho_2$ then , this falls into the definition of **complementary**.
  - If $g(S_2) \neq S_1$ then

    If $g * \rho_1 = \rho_2$ this is the definition of **1-congruent**.

    If $g * \rho_1 = -\rho_2$, this is the definition of **complementary**.
- If for any $g \in \mathcal{E}^+, g(S_1) \neq S_2$ (Lemma 2.2.20):

  This is the definition of **distinct**.

□

Corollary 2.2.23 *Except for a pair of planar surface primitive features,* distinct, 1-congruent, 2-congruent *and* complementary *relationships are mutually exclusive relations between a pair of primitive features.*

Proposition 2.2.24 *If features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ are complementary of each other, where $a(S_1) = S_2, a \in \mathcal{E}^+$, and G_1, G_2 are the symmetry groups of F_1, F_2 respectively, then the two symmetry groups are conjugate via a i.e. $aG_1a^{-1} = G_2$. In particular, if $S_1 = S_2$ then $G_1 = G_2$* **(the necessary condition for surface contact)**.

Proof :

For all $g = ag_1a^{-1} \in aG_1a^{-1}, g(S_2) = ag_1a^{-1}(S_2) = ag_1(S_1) = a(S_1) = S_2$.

For all $(s, v) \in \rho_2$ by definition of complementary features $(s, -v) \in a * \rho_1 = (ag_1a^{-1}a) * \rho_1 = g * (a * \rho_1)$, where $g = ag_1a^{-1} = tr \in aG_1a^{-1}$. Thus $(g^{-1}s, r^{-1}v) \in a * \rho_1$. By the definition of complementary features $(g^{-1}s, r^{-1}v) \in \rho_2$. Then $(s, v) \in g * \rho_2$. Therefore $\rho_2 \subseteq g * \rho_2$.

On the other hand, $\forall(gs, rv) \in g * \rho_2, (s, v) \in \rho_2$. By the definition of complementary features $(s, -v) \in a*\rho_1$. Then $(gs, -rv) \in g(a*\rho_1) = a*\rho_1$. By the definition of complementary features again, $(gs, rv) \in \rho_2$. So $g * \rho_2 \subseteq \rho_2$.

Therefore for all $g \in aG_1a^{-1}, g * \rho_2 = \rho_2$. That is aG_1a^{-1} is a symmetry group for F_2. Hence $aG_1a^{-1} \subseteq G_2$.

Now we need to prove: $G_2 \subseteq aG_1a^{-1}$, i.e. G_2 is a symmetry group of $a(S_1)$.

If $g = tr \in G_2$ then first consider how it acts on the set $g(a(S_1)) = g(S_2) = S_2 = a(S_1)$. Now let us consider how g acts on the orientations. For all $(s, v) \in a * \rho_1, \exists(s, -v) \in \rho_2 = g * \rho_2$, then $(g^{-1}s, r^{-1}v) \in \rho_2 \Rightarrow (g^{-1}s, -r^{-1}v) \in a * \rho_1 \Rightarrow (s, v) \in g(a * \rho_1)$. So $a * \rho_1 \subseteq g(a * \rho_1)$. On the other hand, $\forall(gs, rv) \in g(a * \rho_1), \exists(s, v) \in a * \rho_1 \Rightarrow (s, -v) \in \rho_2 \Rightarrow (gs, -rv) \in g * \rho_2 = \rho_2 \Rightarrow (gs, rv) \in a * \rho_1$. So $g(a * \rho_1) \subseteq a * \rho_1$. One can conclude $g(a * \rho_1) = a * \rho_1$. Therefore $G_2 \subseteq aG_1a^{-1}$.

Hence $G_2 = aG_1a^{-1}$. In case $a = 1, G_1 = G_2$. □

The following lemma shows that for any non-planar primitive features, symmetries for the set-features are the symmetries for the oriented features.

Lemma 2.2.25 *For any non-planar primitive feature $F = (S, \rho)$, if there exists an isometry $g = tr$ such that $g(S) = S$ then $g * \rho = \rho$.*

Proof :

By lemma 2.2.21, $g * \rho = \rho$ or $g * \rho = -\rho$. To prove by contradiction let us assume that $g * \rho = -\rho$. By definition 2.2.19 F is complementary with itself.

Since in Euclidean space a rotation cannot invert more than two independent vectors simultaneously[6] [40, 16], an oriented surface F has to have at most two normals in order for all of its normals to be inverted by a rotation. The only such surface is a planar surface (it is even the only surface with a finite number of normals).

or even a surface which has

Thus F is a planar surface, a contradiction. □

This lemma says that a symmetry for the set of points on a surface is a symmetry for the orientation of the surface as well. This could be seen as a justification for treating an oriented surface as a subset in Euclidean space. Unfortunately, this result does not hold for compound features. Consider the compound feature which is composed of two cylindrical surfaces in case (b) of Figure 1.13, any transformations that interchange the two surfaces (symmetries of the compound feature) will reverse the orientations at each point of the feature.

1.2.3 Symmetry Groups of Compound Features

The following definition and three theorems are from [11]. We shall use these in our proofs.

Definition 2.3.26 *Two sets H, K are* **separated** *if*

$$\bar{H} \cap K = H \cap \bar{K} = \emptyset.$$

Theorem 2.3.27 *A set $M \subset X$ is* **connected** *if and only if M is not the union of two nonempty separated sets.*

Theorem 2.3.28 *For sets, connectivity is preserved by surjective mappings.*

Theorem 2.3.29 *If H and K are separated, then every connected subset M of $H \cup K$ lies either in H or in K.*

Proposition 2.3.30 *Given a compound feature $F = (S, \rho)$ of primitive features $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$ where $F_1, ..., F_n$ are pairwise distinct primitive features with symmetry groups $G_1, ...G_n$ respectively. Then the symmetry group G of F is $G = G_1 \cap ... \cap G_n$.*

[6] If R is a rotation and $\vec{u}, \vec{v}$ are vectors in Euclidean space, then the vector cross product obeys: $R(\vec{u}) X R(\vec{v}) = R(\vec{u} X \vec{v})$

Proof :

Let $g \in G$, then $g(S) = S$. Thus $g(S_1 \cup ... \cup S_n) = g(S_1) \cup ... \cup g(S_n) = S_1 \cup ... \cup S_n$. Then $g(S_i) \subseteq S_1 \cup ... \cup S_n$.

From Lemma 2.2.20 and the definition of distinct features (Definition 2.2.19) we know that $\forall g \in G, g(S_i) = S_i, i = 1...n$.

By Lemma 2.2.25 we have for all the non-planar primitive features $g * \rho_i = \rho_i$. Since $F_1...F_n$ are pairwise distinct there is at most one planar feature whose orientation has to be mapped to itself.

Therefore $g \in G_i$ for $i = 1, ..., n$. Thus $g \in G_1 \cap ... \cap G_n \Rightarrow G \subseteq G_1 \cap ... \cap G_n$.

For all $g \in G_1 \cap ... \cap G_n, g(S) = g(S_1 \cup ... \cup S_n) = g(S_1) \cup ... \cup g(S_n) = S_1 \cup ... \cup S_n = S$ and $g * \rho = g * (\rho_1 \cup ... \cup \rho_n) = g * \rho_1 \cup ... \cup g * \rho_n = \rho_1 \cup ... \cup \rho_n = \rho \Rightarrow g \in G \Rightarrow G_1 \cap ... \cap G_n \subseteq G$.

Therefore $G = G_1 \cap ... \cap G_n$. □

Lemma 2.3.31 *For any pair of primitive features* $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ *where* $S_1 \neq S_2$*, if there exists a* $g \in \mathcal{E}^+$ *such that* $g(S_1 \cup S_2) = S_1 \cup S_2$ *then* $g(S_1) = S_1, g(S_2) = S_2$ *or* $g(S_1) = S_2, g(S_2) = S_1$*.*

Proof :

There are two possibilities for S_1 and S_2:

- $S_1 \cap S_2 = \emptyset$.

  Since $g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = S_1 \cup S_2$, and $g(S_1)$ is a connected subset of $S_1 \cup S_2$ (Theorem 2.3.28), by Theorem 2.3.29 $g(S_1) \subseteq S_1$ or $g(S_1) \subseteq S_2$. If $g(S_1) \subseteq S_1$ then, due to connectivity, $g(S_2) \subseteq S_2$. Since g is a bijection $g(S_1) = S_1, g(S_2) = S_2$. Similarly, $g(S_1) = S_2, g(S_2) = S_1$.

- $S_1 \cap S_2 \neq \emptyset$.

  If there exist open sets $O_1 \subset g(S_1) \cap S_1$ and $O_2 \subset g(S_1) \cap S_2$. Then by Lemma 2.2.20 $g(S_1) = S_1$ and $g(S_1) = S_2$. Thus $S_1 = S_2$, a contradiction. Thus either $g(S_1)$ and S_1 share an open set such that $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$.

Therefore $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$. □

Proposition 2.3.32 *Let a compound feature $F = (S, \rho)$ be composed of a pair of primitive features $F_1 = (S_1, \rho_1)$ and $F_2 = (S_2, \rho_2)$ that are 1-congruent of each other. If G_1, G_2 are the symmetry groups of F_1, F_2 respectively, and G is the symmetry group of F, then $G = G_1 \cap G_2$.*

Proof :
For all $g \in G, g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2)$ and $g * \rho = g * (\rho_1 \cup \rho_2) = g * \rho_1 \cup g * \rho_2$. By Lemma 2.3.31,

- $g(S_1) = S_1, g(S_2) = S_2$:

  If F_1, F_2 are planar features, they have to be parallel planes with their normals pointing to the same direction, i.e. $\rho = \rho_1 = \rho_2$. Thus $g * \rho = \rho \Rightarrow g * \rho_1 = \rho_1$ and $g * \rho_2 = \rho_2$. For non-planar features $g * \rho_1 = \rho_1, g * \rho_2 = \rho_2$ (Lemma 2.2.25).

- $g(S_1) = S_2, g(S_2) = S_1$:

  If $g * \rho_1 = \rho_2$ then F_1, F_2 are 2-congruent; if $g * \rho_1 = -\rho_2$ then F_1, F_2 are complementary; both contradict the fact that F_1, F_2 are 1-congruent.

Then $g \in G_1 \cap G_2$. So we have $G \subseteq G_1 \cap G_2$.

On the other hand, for all $g \in G_1 \cap G_2$, $g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = S_1 \cup S_2 = S$; $g * \rho = g * (\rho_1 \cup \rho_2) = g * \rho_1 \cup g * \rho_2 = \rho_1 \cup \rho_2 = \rho$. Therefore $g \in G \Rightarrow G_1 \cap G_2 \subseteq G$. Thus we conclude $G = G_1 \cap G_2$. □

Proposition 2.3.33 *Let a compound feature $F = (S, \rho)$ be composed of a pair of primitive features $F_1 = (S_1, \rho_1)$ and $F_2 = (S_2, \rho_2)$ that are 2-congruent of each other via g_c. If F_1, F_2 have symmetry groups G_1, G_2 respectively, and G is the symmetry group of F, then $G =< g_c > (G_1 \cap G_2)$, where $< g_c >$ denotes the subgroup of $\mathcal{E}^+$ generated by g_c.*

Proof :
If $g \in G$ then by Lemma 2.3.31 either

- $g(S_1) = S_1$ and $g(S_2) = S_2$:

  By Lemma 2.2.25, taking planar feature case into consideration also, $g * \rho_1 = \rho_1, g * \rho_2 = \rho_2$. Thus $g \in G_1$ and $g \in G_2 \Rightarrow g \in G_1 \cap G_2$; or

- or $g(S_1) = S_2$ and $g(S_2) = S_1 \Rightarrow g^2 = 1$:

  g can be written as $g = g_c g_c^{-1} g$. Let $g_0 = g_c^{-1} g$. $g_0(S_1) = g_c^{-1} g(S_1) = g_c^{-1}(S_2) = S_1, g_0 * \rho_1 = (g_c^{-1} g) * \rho_1 = g_c^{-1} * \rho_2 = g_c * \rho_2 = \rho_1$ (Lemma 2.2.21).

  Therefore $g_0 \in G_1$. Similarly we can prove $g_0 \in G_2$. Thus $g_0 \in G_1 \cap G_2 \Rightarrow g \in < g_c > (G_1 \cap G_2) \Rightarrow G \subseteq < g_c > (G_1 \cap G_2)$;

Therefore $G \subseteq < g_c > (G_1 \cap G_2)$.

On the other hand, if $g \in < g_c > (G_1 \cap G_2)$ then $g = g' g_{12}$ where $g' \in < g_c >$ and $g_{12} \in G_1 \cap G_2$. Then $g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = g' g_{12}(S_1) \cup g' g_{12}(S_2) = g'(S_1) \cup g'(S_2)$. By lemma 2.3.31, either $g'(S_1) \cup g'(S_2) = S_1 \cup S_2 = S$ or $g'(S_1) \cup g'(S_2) = S_2 \cup S_1 = S$. For orientations $g * \rho = g * (\rho_1 \cup \rho_2) = g' g_{12} * \rho_1 \cup g' g_{12} * \rho_2 = g' * \rho_1 \cup g' * \rho_2$. Since $g' \in < g_c >$, by definition of 2-congruent (Definition 2.2.19) either $g' * \rho_1 \cup g' * \rho_2 = \rho_1 \cup \rho_2 = \rho$ or $g' * \rho_1 \cup g' * \rho_2 = \rho_2 \cup \rho_1 = \rho$. Therefore $g \in G \Rightarrow < g_c > (G_1 \cap G_2) \subseteq G$.

Thus we conclude $G = < g_c > (G_1 \cap G_2)$. □

In general, the symmetry group G of a compound feature F can be found from the intersection of the symmetry groups G_i of its primitive features. When 2-congruent features exist, the mappings that flip 2-congruent features in F may also contribute to G. These are new symmetries that do not exist in any individual G_i, but only when all the F_is are considered collectively, and the new group they generated is a discrete group.

Now we can provide some answers to the questions raised earlier, i.e. what is G in expression (1.5) (page 22)? and how do we compute G from G_{11}, G_{12}? From the propositions above we know that when G is the symmetry group of a compound feature F composed of *two* primitive features F_{11} and F_{12} of S_1 or F_{21} and F_{22} of S_2, then if F_{11}, F_{12} (or F_{21} and F_{22}) are distinct or 1-congruent

$$f_1 G f_2^{-1} = f_1 (G_{11} \cap G_{12}) f_2^{-1};$$

and if F_{11}, F_{12} (or F_{21} and F_{22}) are 2-congruent via g_c

$$f_1 G f_2^{-1} = f_1 < g_c > (G_{11} \cap G_{12}) f_2^{-1}.$$

1.3 Summary

In this chapter we have established the formalism of expressing relative positions of contacting solids using symmetry groups. We have also established the basic vocabulary in terms of oriented surface primitives. Special attention is given to surface contacts among solids, and the computational characterization of the symmetry group of multiple contacting surfaces.

Although surface contact is only one of the three cases (surface, line, point) among solid contacts, it is significant in practice: "Indeed, surface contacts are simplest to execute (by a robot), because: (1) there are less degrees of freedom left than for point or line contacts, so you have to give less motion specification information. (2) a surface contact is more stable: if you change the contact force and/or moment a little bit, the relative position of the two contacting bodies remains unchanged, IF there is a compliance in the system which can 'absorb' the force/moment differences."[7]. Furthermore even in the most general form of solid contact expressed as group products, the intersection of subgroups are required. The intersection of symmetry groups of the primitive surface features is one of the crucial operations in determining the relative motions of contacting solids.

We have proven:

- distinct, 1-congruent, 2-congruent and complementary relationship characterizations of oriented surface features are complete and mutually exclusive;
- the symmetry group G of a compound feature F, that is composed of a set of oriented primitive features F_i with symmetry group G_i, depends on the relationships between those primitive features, and can be expressed as
  - if F_is are pairwise *distinct*, or *1-congruent* then $G = \cap G_i$;
  - if F_is are 2-congruent and $i = 2$, then $G =< g_{ij} > (\cap G_i)$ where g_{ij} is a transformation that switches a pair of oriented features F_i, F_j in F and is a symmetry of F, and $< g_{ij} >$ indicates the group generated by such a g_{ij}.

- a constructive framework for finding the symmetry group of a collection of oriented surfaces from individual surface symmetry groups.

These results lay out a realistic and precise group theoretic framework for characterizing surfaces of solids and capture the very nature of surface contact — the state of being complementary.

We raised these questions:

1. Given two solids S_1, S_2, what is the relative location of the two under n surface contact (n primitive features from each side, Figure 1.9)?

2. Given two solids S_1, S_2, what is the relative location of the two under n general contact (Figure 1.9)?

3. Given m solids in a chaining general contact (Figure 1.10), what is the relative location of the mth solid with respect to the first solid?

The hypothesis is that using a group theoretical formalization of the oriented surfaces of solids in contact, these questions can be answered automatically and precisely. What we have achieved so far includes a thorough understanding of surface contact among solids, a precise expression and constructive proven results for finding the symmetry groups in case (1); for cases (2) and (3), we are able to construct the precise expressions for relative positions in terms of contacting surface symmetry groups (equations (1.7), (1.8)).

In order to make the group formalization computable, we have to solve at least these two computational problems:

1. How to denote symmetry groups, which can be finite, infinite, discrete or continuous, on computers?

2. How to do intersections of subgroups of $\mathcal{E}^+$ on computers efficiently?

The answers to these questions are to be found in the Chapter **??**: A Geometric Approach for Groups, where an algorithm to compute the symmetry groups in case (1) is proven to be correct and a computer implementation is given.

The applications of such a formalism are demonstrated in Chapter **??**: "A Computational Representation for Assembly", where the question that started this book: **how to describe contacts between solids to a robot?** will be answered.

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