15-858 APRIL 17, 2007

LECTURE 12 (with answers)

MATRIX-ANALYTIC METHODS

(a.k.a. Matrix-Geometric methods)

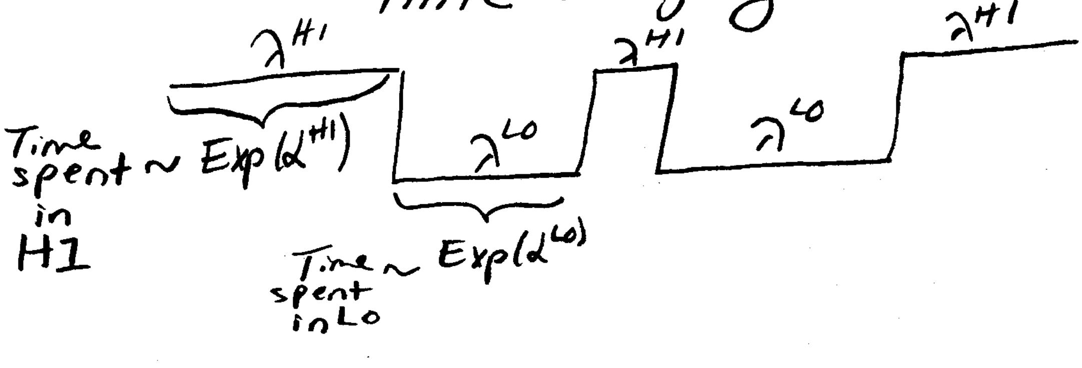
I. MOTIVATION

- Finite-state Markov Chains are easy to solve
- Infinite-state Markov Chains are easy too, if have 1 row & repeat e.g. M/M/1
- But what do you do when have >1 row and infinite states?

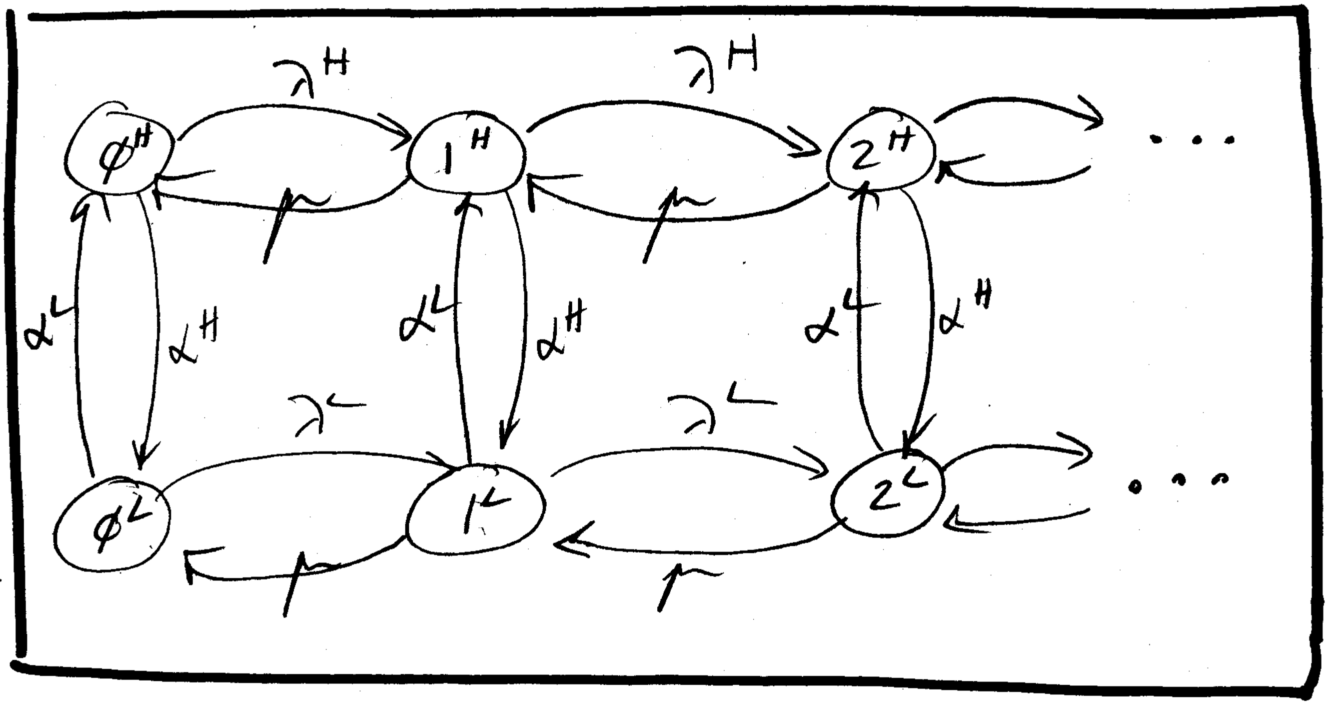
II. EXAMPLE – $M_t/M/1$

M_t: Time-varying arrival rate

M: Exp(μ)



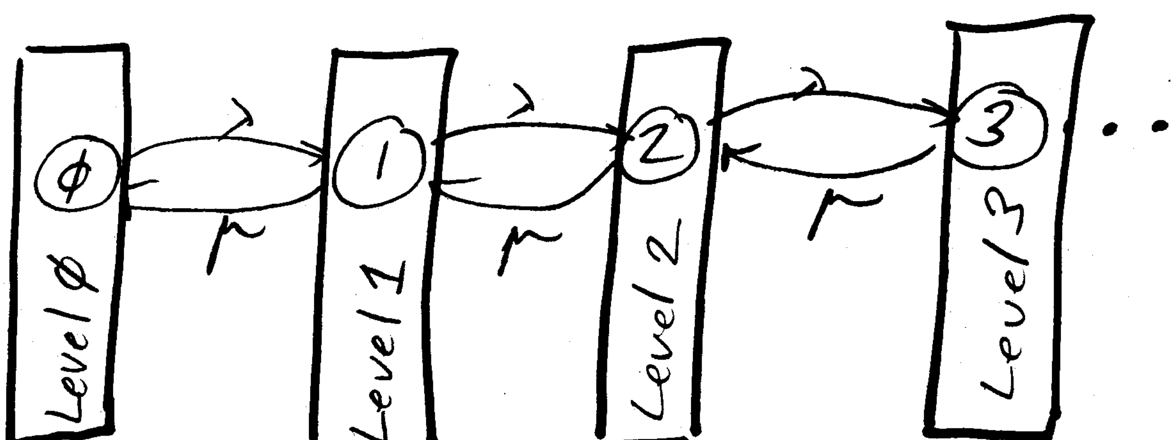
FILL IN CTMC REPRESENTING $M_t/M/1$ MODEL:



NOTE: ① REPEATING STRUCTURE

② CAN WE SOLVE RECURSIVELY LIKE IN M/M/1?

III. BACK TO M/M/1.



- Write balance eqns in this form

$$\vec{\pi} \cdot Q = \vec{\phi}$$

$(\pi_\phi, \pi_1, \pi_2, \ldots)$

"GENERATOR MATRIX"

What is Q?

$$\boxed{Q = \begin{array}{c|c} & \begin{array}{ccccc} \phi & 1 & 2 & 3 & \cdots \end{array} \\ \begin{array}{c} \phi \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \mu & -(\lambda+\mu) & \lambda & \\ & & \mu & -(\lambda+\mu) & \\ & & & \mu & \ddots \end{bmatrix} \end{array}}$$

We also need normalization eqn:
Write as product of 2 vectors:

$$\boxed{\vec{\pi} \cdot \vec{e} = 1}$$

IV Q AND ITS SUBMATRICES

What is Q for the $M_t/M/1$?

$$\vec{\pi} \cdot Q = \vec{\phi}$$

$$
Q = \begin{array}{c} \\ \phi^H \\ \phi^L \\ 1^H \\ 1^L \\ 2^H \\ 2^L \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \phi^H & \phi^L & 1^H & 1^L & 2^H & 2^L & & \end{array} \\
\left[\begin{array}{cc|cc|cc|c}
-(\lambda^H+\alpha^H) & \alpha^H & \lambda^H & \phi & & & \\
\alpha^L & -(\lambda^L+\alpha^L) & \phi & \lambda^L & & & \\
\hline
\mu & \phi & -(\lambda^H+\alpha^H+\mu) & \alpha^H & \lambda^H & \phi & \\
\phi & \mu & \alpha^L & -(\lambda^L+\alpha^L+\mu) & \phi & \lambda^L & \\
\hline
 & & \mu & \phi & -(\lambda^H+\alpha^H+\mu) & \alpha^H & \\
 & & \phi & \mu & \alpha^L & -(\lambda^L+\alpha^L+\mu) & \\
\hline
 & & & & & & \ddots
\end{array}\right]
\end{array}
$$

<u>Notation</u>: $\vec{\pi}_j = (\pi_j^{H}, \pi_j^{L})$

Backward

$$B = \begin{bmatrix} \mu & \phi \\ \phi & \mu \end{bmatrix}$$

Local

$$L = \begin{bmatrix} -(\lambda^H+\alpha^H+\mu) & \alpha^H \\ \alpha^L & -(\lambda^L+\alpha^L+\mu) \end{bmatrix}$$

Forward

$$F = \begin{bmatrix} \lambda^H & \phi \\ \phi & \lambda^L \end{bmatrix}$$

$$L_0 = \begin{bmatrix} -(\lambda^H+\alpha^H) & \alpha^H \\ \alpha^L & -(\lambda^L+\alpha^L) \end{bmatrix} = \text{Local transitions for initial part}$$

$$Q = \begin{array}{c} \\ \phi^H \\ \phi^L \\ 1^H \\ 1^L \\ 2^H \\ 2^L \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} \phi^H & \phi^L & 1^H & 1^L & 2^H & 2^L & \cdots \end{array} \\ \left[\begin{array}{c:c:c:c} L_\phi & F & & \\ \hdashline B & L & F & \\ \hdashline & B & L & F \\ \hdashline & & B & L \end{array}\right] \end{array}$$

Fill in matrix names

Rewrite balance eqns: $\vec{\pi}\cdot Q = \vec{\phi}$ as matrix eqns.

$$\vec{\pi}_\phi \cdot L_\phi + \vec{\pi}_1 \cdot B = \vec{\phi}$$
$$\vec{\pi}_\phi \cdot F + \vec{\pi}_1 \cdot L + \vec{\pi}_2 \cdot B = \vec{\phi}$$
$$\vec{\pi}_1 \cdot F + \vec{\pi}_2 \cdot L + \vec{\pi}_3 \cdot B = \vec{\phi}$$
$$\vec{\pi}_2 \cdot F + \vec{\pi}_3 \cdot L + \vec{\pi}_4 \cdot B = \vec{\phi}$$

V. SOLVING FOR LIMITING PROBABILITIES

GUESS: $\vec{\pi}_i = \vec{\pi}_\phi \cdot R^i$

$\vec{\pi}_i = (\pi_i^H, \pi_i^L)$

Substitute above "guess" into balance eqns:

$$\boxed{\begin{aligned}
&\vec{\pi}_\phi \cdot L_\phi + \vec{\pi}_\phi RB = \vec{\phi} &&\Rightarrow \vec{\pi}_\phi (L_\phi + RB) = \vec{\phi} \\
&\vec{\pi}_\phi \cdot F + \vec{\pi}_\phi RL + \vec{\pi}_\phi \cdot R^2 B = \vec{\phi} &&\Rightarrow \vec{\pi}_\phi (F + RL + R^2 B) = \vec{\phi} \\
&\vec{\pi}_1 \cdot F + \vec{\pi}_1 \cdot RL + \vec{\pi}_1 \cdot R^2 B = \vec{\phi} &&\Rightarrow \vec{\pi}_1 (F + RL + R^2 B) = \vec{\phi} \\
&\vec{\pi}_2 \cdot F + \vec{\pi}_2 \cdot RL + \vec{\pi}_2 \cdot R^2 B = \vec{\phi} &&\Rightarrow \vec{\pi}_2 (F + RL + R^2 B) = \vec{\phi}
\end{aligned}}$$

Common Portion: $\Rightarrow F + RL + R^2 B = \phi$

To get $\mathbf{R}$

$$\mathcal{F} + RL + R^2 B = \phi$$

$$\Rightarrow \quad RL = -(R^2 B + \mathcal{F})$$

$$\Rightarrow \quad R = -(R^2 B + \mathcal{F}) L^{-1}$$

Solve for $\mathbf{R}$ BY ITERATNG:

$$R(\phi) = \mathbf{0}$$ } ← matrix of appropriate dimension

while $\| R(n+1) - R(n) \| > \varepsilon$,

$$R(n+1) = -(R^2(n) B + F) L^{-1}$$

Keep iterating until get R

now: $\vec{\pi}_i = \vec{\pi}_\phi \cdot R^i$

satisfies balance eqns.

To get $\vec{\pi}_\phi = (\pi_\phi^{HI}, \pi_\phi^{LO})$

Fill in first Matrix Balance Eqn involving $\vec{\pi}_\phi$

$$\boxed{\vec{\pi}_\phi (L_\phi + RB) = \vec{\phi}}$$

Rewrite normalizing eqn $\vec{\pi} \cdot \vec{e} = 1$ in terms of R and $\vec{\pi}_\phi$

$$\sum_{i=\phi}^{\infty} \vec{\pi}_i \cdot \vec{e} = 1$$

$$\sum_{i=\phi}^{\infty} \vec{\pi}_\phi \cdot R^i \cdot \vec{e} = 1$$

$$\vec{\pi}_\phi \left(\sum_{i=\phi}^{\infty} R^i \right) \cdot \vec{e} = 1$$

$$\boxed{\vec{\pi}_\phi \left[(I-R)^{-1} \right] \cdot \vec{e} = 1}$$

Let $\Phi = L_\phi + RB$

$$(\pi_\phi^{HI}, \pi_\phi^{LO}) \cdot \begin{pmatrix} \Phi_{00} & \Phi_{01} \\ \Phi_{10} & \Phi_{11} \end{pmatrix} = \begin{pmatrix} \phi \\ \phi \end{pmatrix}$$

REPLACE WITH: $\boxed{1}$

REPLACE WITH: $\boxed{(I-R)^{-1}\vec{e}}$

SOLVE ABOVE FOR $\vec{\pi}_\phi = (\pi_\phi^{HI}, \pi_\phi^{LO})$. (5

VI. PERFORMANCE METRICS

- Derive closed-form expression for $E[N_s]$ in terms of only $\vec{\pi}_\phi$ and R:

$$E[N_s] = \sum_{i=\phi}^{\infty} i \cdot \vec{\pi}_i \cdot \vec{e}$$

$$= \sum_{i=\phi}^{\infty} i \cdot \vec{\pi}_\phi \cdot R^i \cdot \vec{e}$$

$$= \vec{\pi}_\phi \sum_{i=\phi}^{\infty} i \cdot R^{i-1} \cdot R \cdot \vec{e}$$

$$= \vec{\pi}_\phi \frac{d}{dR}\left(\sum_{i=\phi}^{\infty} R^i\right) \cdot R \cdot \vec{e}$$

$$= \vec{\pi}_\phi \frac{d}{dR} (I-R)^{-1} \cdot \vec{e}$$

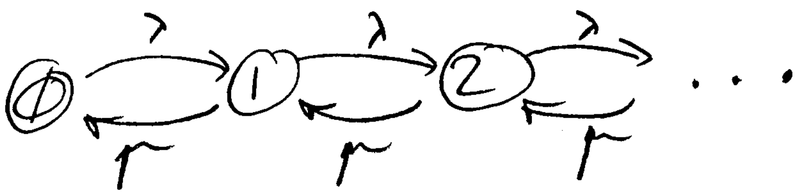
$$= \vec{\pi}_\phi (-1)(I-R)^{-2} \cdot (-1) \cdot R \cdot \vec{e}$$

$$\boxed{E[N_s] = \vec{\pi}_\phi \cdot (I-R)^{-2} \cdot R \cdot \vec{e}}$$

- $$\boxed{E[T_s] = \frac{1}{\lambda_{AVG}} \vec{\pi}_\phi \cdot (I-R)^{-2} \cdot R \cdot \vec{e}}$$

$$\lambda_{AVG} = \frac{1/\alpha^H \cdot \lambda^H + 1/\alpha^L \cdot \lambda^L}{1/\alpha^H + 1/\alpha^L}$$

VII Back to M/M/1

- 

$\vec{\pi} \cdot Q = \vec{0}$

$$Q = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \mu & -(\lambda+\mu) & \lambda & \\ & & \mu & -(\lambda+\mu) & \\ & & & & \end{bmatrix}$$

- $B = \boxed{\mu}$ $\quad L = \boxed{-(\lambda+\mu)}$ $\quad F = \boxed{\lambda}$ $\quad L_0 = \boxed{-\lambda}$

GUESS: $\pi_i = \pi_0 \cdot R^i$

FROM BAL EQNS: $F + RL + R^2 B = 0$

WHAT DOES THIS SAY ABOUT R?

$$\boxed{\begin{gathered} \lambda + R(-(\lambda+\mu)) + R^2\mu = 0 \\ \Rightarrow R = \rho \end{gathered}}$$

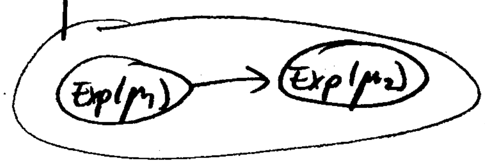
- $\vec{\pi} \cdot \vec{e} = 1 \Rightarrow \boxed{\pi_0 = (1-\rho)}$

VIII MORE COMPLEX CHAINS

Sometimes repeating portion only starts after level M.

$$M^* / G / 1 / FCFS$$

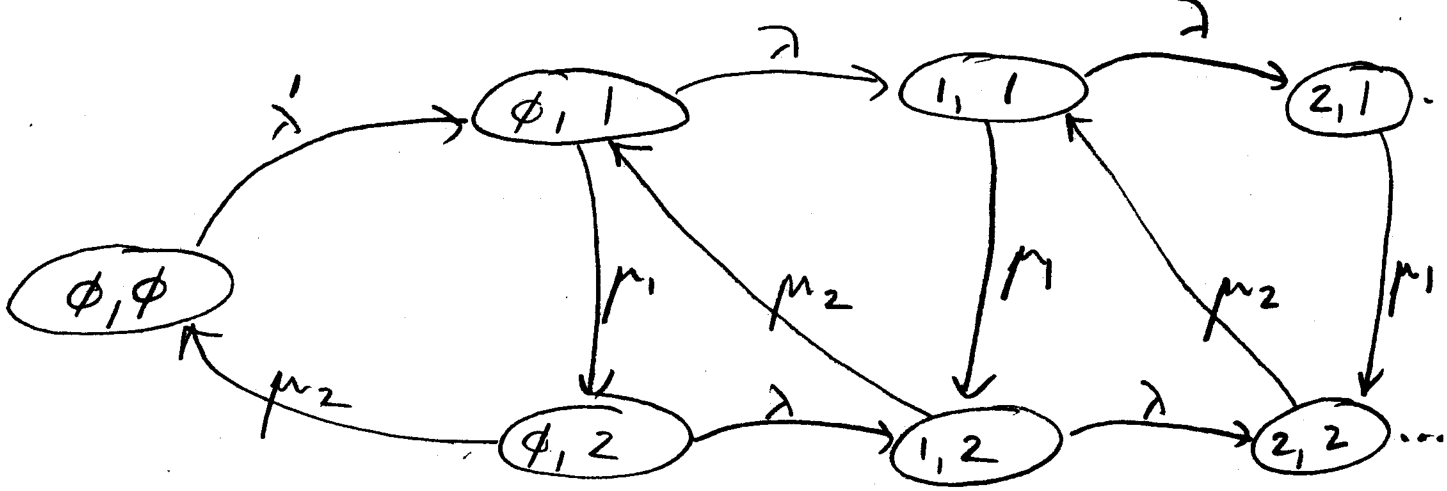
Exp(λ) normally but Exp(λ') if system empty.



State = (i, s)

jobs in queue only (not in service)

PHASE OF Job in service



$$
Q = \begin{array}{c} \\ (\phi,\phi) \\ (\phi,1) \\ (\phi,2) \\ (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \\ (3,1) \\ (3,2) \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} (\phi,\phi) & (\phi,1) & (\phi,2) & (1,1) & (1,2) & (2,1) & (2,2) & (3,1) & (3,2) \end{array} \\
\left[\begin{array}{ccc|cc|cc|cc}
-\lambda' & \lambda' & \phi & \phi & \phi & & & & \\
\phi & -a_1 & \mu_1 & \lambda & \phi & & & & \\
\mu_2 & \phi & -a_2 & \phi & \lambda & & & & \\
\hline
\phi & \phi & \phi & -a_1 & \phi & \lambda & \phi & & \\
\phi & \mu_2 & \phi & \phi & -a_2 & \phi & \lambda & & \\
\hline
 & & & a & \phi & -a_1 & \phi & \lambda & \phi \\
 & & & \mu_2 & \phi & \phi & -a_2 & \phi & \lambda \\
\hline
 & & & & & 0 & \phi & -a_1 & \phi \\
 & & & & & \mu_2 & \phi & \phi & -a_2
\end{array}\right]
\end{array}
$$

where $a_1 = \lambda + \mu_1$

$a_2 = \lambda + \mu_2$

$$
Q = \begin{array}{c} \\ 3 \\ 2 \\ 2 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{cccc} 3 & 2 & 2 & 2 \end{array} \\
\left[\begin{array}{c|c|c|c}
L_\phi & F_\phi & & \\
\hline
B_\phi & L & F & \\
\hline
 & B & L & F \\
\hline
 & & B & L
\end{array}\right]
\end{array}
$$

$$\vec{\Pi}_\phi = \left(\Pi_{(\phi,\phi)}, \Pi_{(\phi,1)}, \Pi_{(\phi,2)}\right)$$

$$\vec{\Pi}_i = \left(\Pi_{(i,1)}, \Pi_{(i,2)}\right)$$

Balance Eqns

$$\boxed{\begin{aligned}
&\vec{\pi}_\phi L_\phi + \vec{\pi}_1 B_\phi = \vec{\phi} \\
&\vec{\pi}_\phi F_\phi + \vec{\pi}_1 L + \pi_2 B = \vec{\phi} \\
\hline
&\vec{\pi}_1 F + \vec{\pi}_2 L + \vec{\pi}_3 B = \vec{\phi} \\
&\vec{\pi}_2 F + \vec{\pi}_3 L + \vec{\pi}_4 B = \vec{\phi} \\
&\vec{\pi}_3 F + \vec{\pi}_4 L + \vec{\pi}_5 B = \vec{\phi} \\
&\qquad \vdots
\end{aligned}}$$

<u>**GUESS:**</u> $\vec{\pi}_{M+k} = \vec{\pi}_M \cdot R^k$

In this problem, $\vec{\pi}_M = \vec{\pi}_{\boxed{1}}$.

Balance Eqns incorporating guess:

$$\boxed{\begin{array}{l}\vec{\pi}_\phi L_\phi + \pi_1 B_\phi = \vec{\phi} \\ \vec{\pi}_\phi F_\phi + \vec{\pi}_1 L + \vec{\pi}_1 R B = \vec{\phi} \\ \hline \vec{\pi}_1 (F + RL + R^2 B) = \vec{\phi} \\ \vec{\pi}_2 (F + RL + R^2 B) = \vec{\phi} \\ \vec{\pi}_3 (F + RL + R^2 B) = \vec{\phi}\end{array}}$$

To determine R, iterate over common portion:

$$\boxed{\begin{array}{l}F + RL + R^2 B = \phi \\ R_{(n+1)} = -(R^2_{(n)} B + F) L^{-1}\end{array}}$$

To determine initial portion:

$$(\vec{\pi}_\phi \quad \vec{\pi}_1)\begin{bmatrix} L_\phi & F_\phi \\ B_\phi & L+RB \end{bmatrix} = (\phi,\phi,\phi,\phi,\phi)$$

replace by $\boxed{1}$

Replace 1st column by:

$$\boxed{\begin{bmatrix} \vec{e} \\ (I-R)^{-1}\cdot\vec{e} \end{bmatrix}}$$

$$\vec{\pi}_\phi\cdot\vec{e} + \sum_{i=1}^{\infty}\pi_i\vec{e} = 1$$

$$\vec{\pi}_\phi\cdot\vec{e} + \pi_1\sum_{i=\phi}^{\infty}R^i\cdot\vec{e} = 1$$

$$\vec{\pi}_\phi\cdot\vec{e} + \pi_1(I-R)^{-1}\cdot\vec{e} = 1$$