LECTURE 12

MATRIX-ANALYTIC METHODS

(a.k.a. Matrix-Geometric methods)

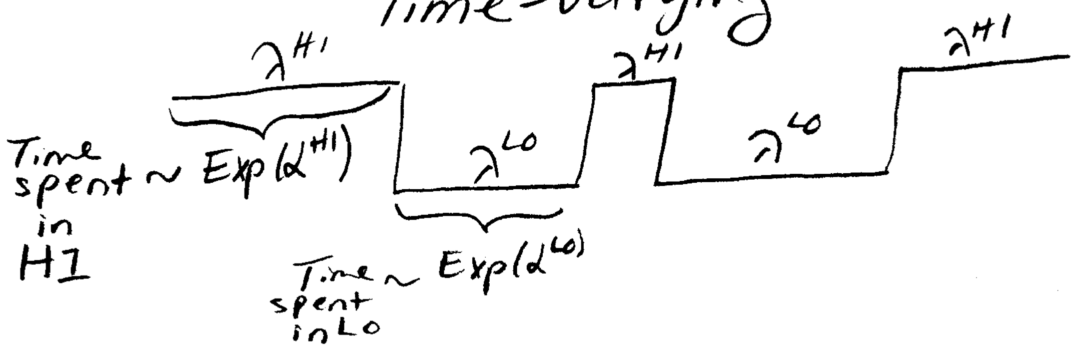
I. MOTIVATION

- Finite-state Markov Chains are easy to solve
- Infinite-state Markov Chains are easy too, if have 1 row ; repeat e.g. M/M/1
- But what do you do when have > 1 row and infinite states?

II. EXAMPLE – $M_t/M/1$

M_t: Time-varying arrival rate

$M/1$: $Exp(\mu)$

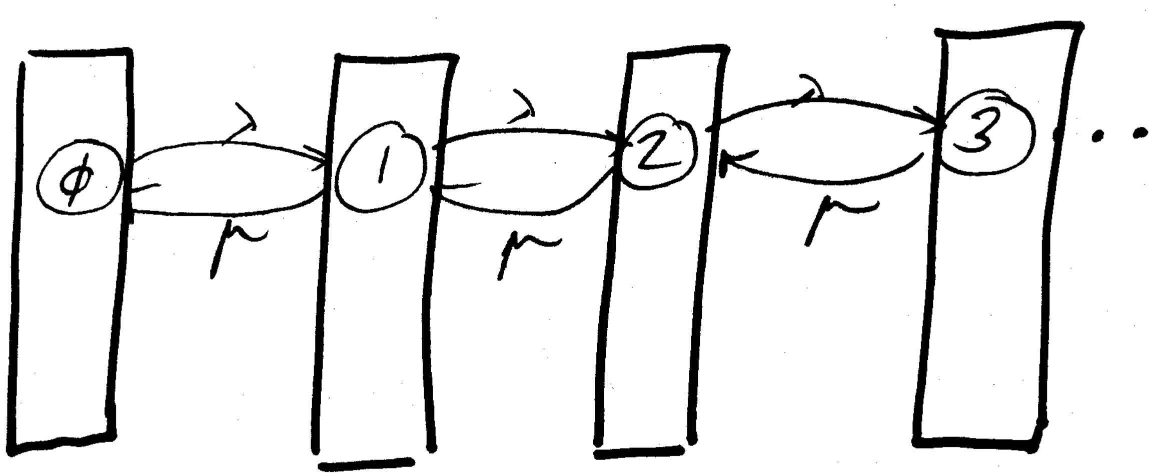


FILL IN CTMC REPRESENTING $M_t/M/1$ MODEL:

NOTE: ① REPEATING STRUCTURE

② CAN WE SOLVE RECURSIVELY LIKE IN $M/M/1$?

III. BACK TO $M/M/1$.



- Write balance eqns in this form

$$\underset{(\Pi_\emptyset, \Pi_1, \Pi_2, \ldots)}{\vec{\Pi}} \cdot Q = \vec{\emptyset}$$

"GENERATOR MATRIX"

<u>What is Q?</u>

$$Q = \begin{array}{c} \\ \emptyset \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \emptyset & 1 & 2 & 3 & \cdots \end{array} \\ \left[\begin{array}{ccccc} \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ddots \end{array} \right] \end{array}$$

We also need normalization eqn:
Write as product of 2 vectors:

IV Q AND ITS SUBMATRICES

What is Q for the $M_t/M/1$?

$$\vec{\pi} \cdot Q = \vec{0}$$

$$Q = \begin{array}{c} \\ 0^H \\ 0^L \\ 1^H \\ 1^L \\ 2^H \\ 2^L \\ \\ \end{array} \begin{array}{c} \begin{array}{cc|cc|cc|c} 0^H & 0^L & 1^H & 1^L & 2^H & 2^L & \end{array} \\ \left[\begin{array}{cc|cc|cc|c} \quad & \quad & \quad & \quad & \quad & \quad & \\ & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ \hline & & & & & & \ddots \end{array} \right] \end{array}$$

<u>Notation</u>: $\vec{\pi}_j = (\pi_j^{H}, \pi_j^{L})$

Backward

$B = \begin{bmatrix} \qquad \end{bmatrix}$

Local

$L = \begin{bmatrix} \qquad \end{bmatrix}$

Forward

$F = \begin{bmatrix} \qquad \end{bmatrix}$

$$L_0 = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} = \text{Local transitions for initial part}$$

$$Q = \begin{array}{cc} & \begin{array}{ccccccc} 0^H & 0^L & 1^H & 1^L & 2^H & 2^L & \cdots \end{array} \\ \begin{array}{c} 0^H \\ 0^L \\ 1^H \\ 1^L \\ 2^H \\ 2^L \\ \vdots \end{array} & \left[\begin{array}{cc|cc|cc|c} \quad & \quad & \quad & \quad & \quad & \quad & \quad \\ & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ \hline & & & & & & \end{array}\right] \end{array}$$

Fill in matrix names

Rewrite balance eqns: $\vec{\pi} \cdot Q = \vec{0}$ as matrix eqns.

$$\boxed{\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ \\ \\ \\ \end{array}}$$

V. SOLVING FOR LIMITING PROBABILITIES

GUESS: $\vec{\pi}_i = \vec{\pi}_\phi \cdot R^i$

$$\vec{\pi}_i = (\pi_i^H, \pi_i^L)$$

Substitute above "guess" into balance eqns:

Common Portion: $\Rightarrow$ $F + RL + R^2 B = \phi$

To get R

$$F + RL + R^2B = \phi$$

$$\Rightarrow \quad RL = -(R^2B + F)$$

$$\Rightarrow \quad R = -(R^2B + F)L^{-1}$$

Solve for $\mathbf{R}$ BY ITERATNG:

$$R(\phi) = \mathbf{0}$$ ← matrix of appropriate dimension

while $\|R(n+1) - R(n)\| > \varepsilon$,

$$R(n+1) = -(R^2(n)B + F)L^{-1}$$

Keep iterating until get R

now: $\vec{\pi}_i = \vec{\pi}_\phi \cdot R^i$

satisfies balance eqns.

To get $\vec{\pi}_\phi = (\pi_\phi^{HI}, \pi_\phi^{LO})$

Fill in first Matrix Balance Eqn Involving $\vec{\pi}_\phi$

Rewrite normalizing eqn $\vec{\pi} . \vec{e} = 1$ in terms of R and $\vec{\pi}_\phi$

Let $\Phi = L_\phi + RB$

$$(\pi_\phi^{HI}, \pi_\phi^{LO}) \cdot \begin{pmatrix} \Phi_{00} & \Phi_{01} \\ \Phi_{10} & \Phi_{11} \end{pmatrix} = \begin{pmatrix} \phi \\ \phi \end{pmatrix}$$

REPLACE WITH:

REPLACE WITH:

SOLVE ABOVE FOR $\vec{\pi}_\phi = (\pi_\phi^{HI}, \pi_\phi^{LO})$.

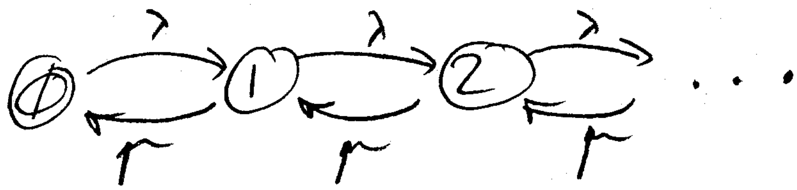
VI. PERFORMANCE METRICS

Derive closed-form expression for $E[N_s]$ in terms of only $\vec{\pi}_\phi$ and R:

$E[N_s] =$

$E[T_s] =$

VII Back to M/M/1



$\vec{\pi} \cdot Q = \vec{0}$

$Q =$

$B =$ $L =$ $F =$ $L_0 =$

GUESS : $\pi_i = \pi_0 \cdot R^i$

FROM BAL EQNS : $F + RL + R^2 B = 0$

WHAT DOES THIS SAY ABOUT R?

$\vec{\pi} \cdot \vec{e} = 1 \Rightarrow \pi_0 =$

VIII More complex chains

Sometimes repeating portion only starts after Level M.

$M^* / G / 1 / FCFS$

$Exp(\lambda)$ nomally but $Exp(\chi)$ if system empty.

$Exp(\mu_1) \rightarrow Exp(\mu_2)$

State $= (i, s)$

i: # jobs in queue only (not in service)

s: PHASE OF Job in service

DRAW THE MARKOV CHAIN:

$$Q = \begin{array}{c c}
 & \begin{array}{ccccccccc} (\phi,\phi) & (\phi,1) & (\phi,2) & (1,1) & (1,2) & (2,1) & (2,2) & (3,1) & (3,2) \end{array} \\
\begin{array}{c} (\phi,\phi) \\ (\phi,1) \\ (\phi,2) \\ (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \\ (3,1) \\ (3,2) \end{array} &
\left[\begin{array}{ccc|cc|cc|cc}
\; & \; & \; & \; & \; & \; & \; & \; & \; \\
 & & & & & & & & \\
 & & & & & & & & \\ \hline
 & & & & & & & & \\
 & & & & & & & & \\ \hline
 & & & & & & & & \\
 & & & & & & & & \\ \hline
 & & & & & & & & \\
 & & & & & & & &
\end{array}\right]
\end{array}$$

where $a_1 = \lambda + \mu_1$

$a_2 = \lambda + \mu_2$

$$Q = \begin{array}{c c}
 & \begin{array}{cccc} 3 & 2 & 2 & 2 \end{array} \\
\begin{array}{c} 3 \\ 2 \\ 2 \\ 2 \end{array} &
\left[\begin{array}{c|c|c|c}
L_\phi & F_\phi & & \\ \hline
B_\phi & L & F & \\ \hline
 & B & L & F \\ \hline
 & & B & L
\end{array}\right]
\end{array}$$

$$\vec{\Pi}_\phi = \left(\Pi_{(\phi,\phi)}, \Pi_{(\phi,1)}, \Pi_{(\phi,2)} \right)$$

$$\vec{\Pi}_i = \left(\Pi_{(i,1)}, \Pi_{(i,2)} \right)$$

Balance Eqns

GUESS: $\vec{\Pi}_{M+k} = \vec{\Pi}_M \cdot R^k$

In this problem, $\vec{\Pi}_M = \vec{\Pi}_{\square}$.

Balance Eqns incorporating guess:

To determine R, iterate over common portion:

To determine initial portion:

$$(\vec{\Pi}_\phi \quad \vec{\Pi}_1)\begin{bmatrix} L_\phi & F_\phi \\ B_\phi & L+RB \end{bmatrix} = (\phi,\phi,\phi,\phi,\phi)$$

replace by

Replace 1st column by:

Homework: Practice with Matrix Analytic Method

Use the matrix analytic method from lecture to analyze the M/PH/1 queue. The arrival process is Poisson(λ). The service time distribution is *phase-type* with 2 stages. Specifically, the service time distribution may be described as follows:

Start with an Exponential(μ_1). After that, with probability $1-p$, do nothing, but with probability p, add on another Exponential(μ_2).

To make this concrete, I will ask you to solve the problem for the case where $\lambda = 1$, $\mu_1 = 2$, $\mu_3 = 3$, and $p = .4$. I highly recommend using Matlab to solve this problem!

Here are some steps to help you:

1. Define a state space.
2. Draw out the Markov Chain.
3. Write out the generator matrix Q (it's infinite, so a portion will do).
4. Write out the matrices denoted by: L_0, F_0, B_0, F, L, B. [Check: L should be a 2x2 matrix].
5. Write out the balance equations and make the appropriate guess for the repeating part of the limiting probabilities.
6. Solve for the matrix R. [Check: R should be a 2x2 matrix].
7. Use the initial (non-repeating) balance equations together with the normalization constraint to solve for the initial limiting probabilities. Remember that these may be vectors. [Check: Compute the probability of there being zero jobs in the system. This should equal $1 - \rho$]. [Hint: When I did this, I found that the probability that there are zero jobs in the system was: .3669 The total probability that there is one job in the system is: .2446]
8. At this point you should have all the limiting probabilities. Use these to compute $E[N_S]$ and $E[T_S]$.
9. Check your answer by applying the P-K formula for the M/G/1 queue.