Solving Graph Problems with Boolean Methods

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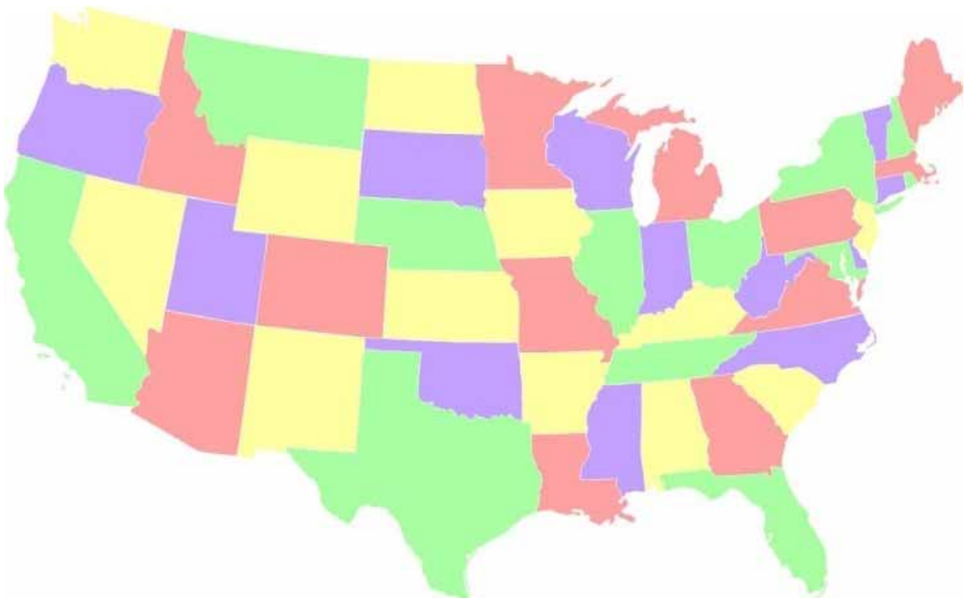
The Graph Coloring Problem

Color the regions of a planar graph

- Use only 4 colors
- No two adjacent regions can have the same color

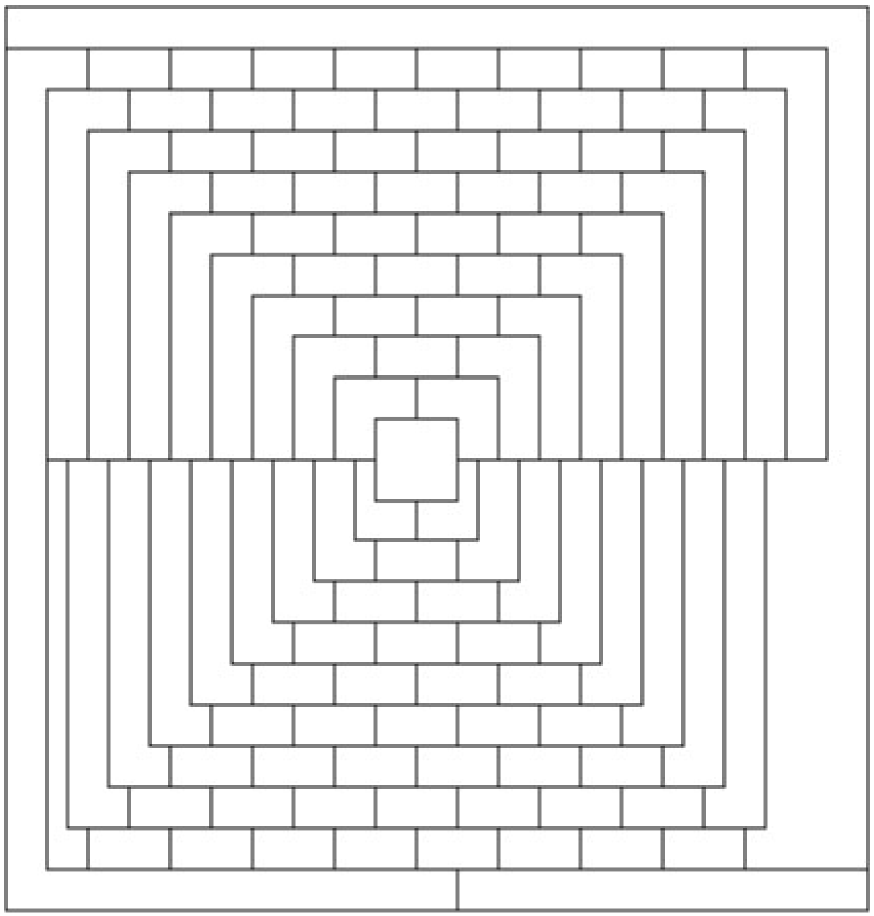


Example: Color the states of a U.S. map

The MacGregor Graph

Scientific American, April 1975

- Said to be proof that some planar graphs could not be colored with just 4 colors
- An April-fool's joke, but still difficult to solve by hand



Boolean SAT Solvers

What They Do

- Express problem as a set of constraints
- Search for solution that satisfies all constraints

Encoding Graph Coloring with SAT

- Encode each region with two 0/1-valued variables:
  - 00 Blue
  - 01 Green
  - 10 Red
  - 11 Yellow
- For each adjacent region, require at least one of the corresponding variables to have opposite values

Encoding Coloring Constraints

a_i, b_i	a_j, b_j

- **Encode region *i* with variables a_i, b_i**
- **For adjacent regions *i* and *j*, want:**

  $a_i \neq a_j \vee b_i \neq b_j$
- **Clausal form (and of or's):**

$c = 0 \Rightarrow a_i \neq a_j$

$$\boxed{(c \vee a_i \vee a_j) \wedge (c \vee \neg a_i \vee \neg a_j)} \wedge \boxed{(\neg c \vee b_i \vee b_j) \wedge (\neg c \vee \neg b_i \vee \neg b_j)}$$

$c = 1 \Rightarrow b_i \neq b_j$

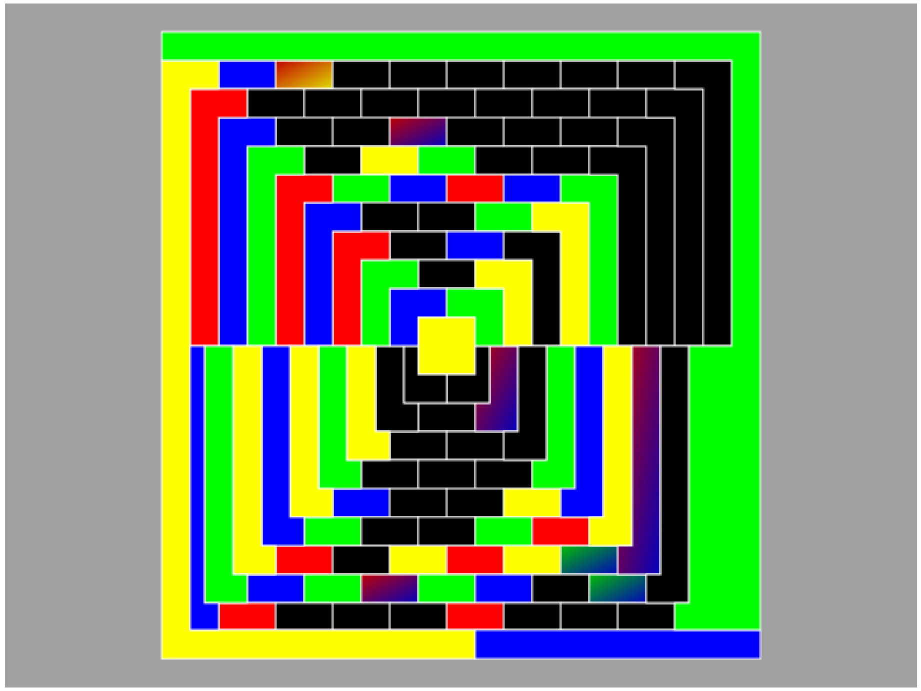
The ZChaff SAT Solver

- From Princeton University
- Algorithm by Davis Putnam Logemann & Loveland
- With many refinements

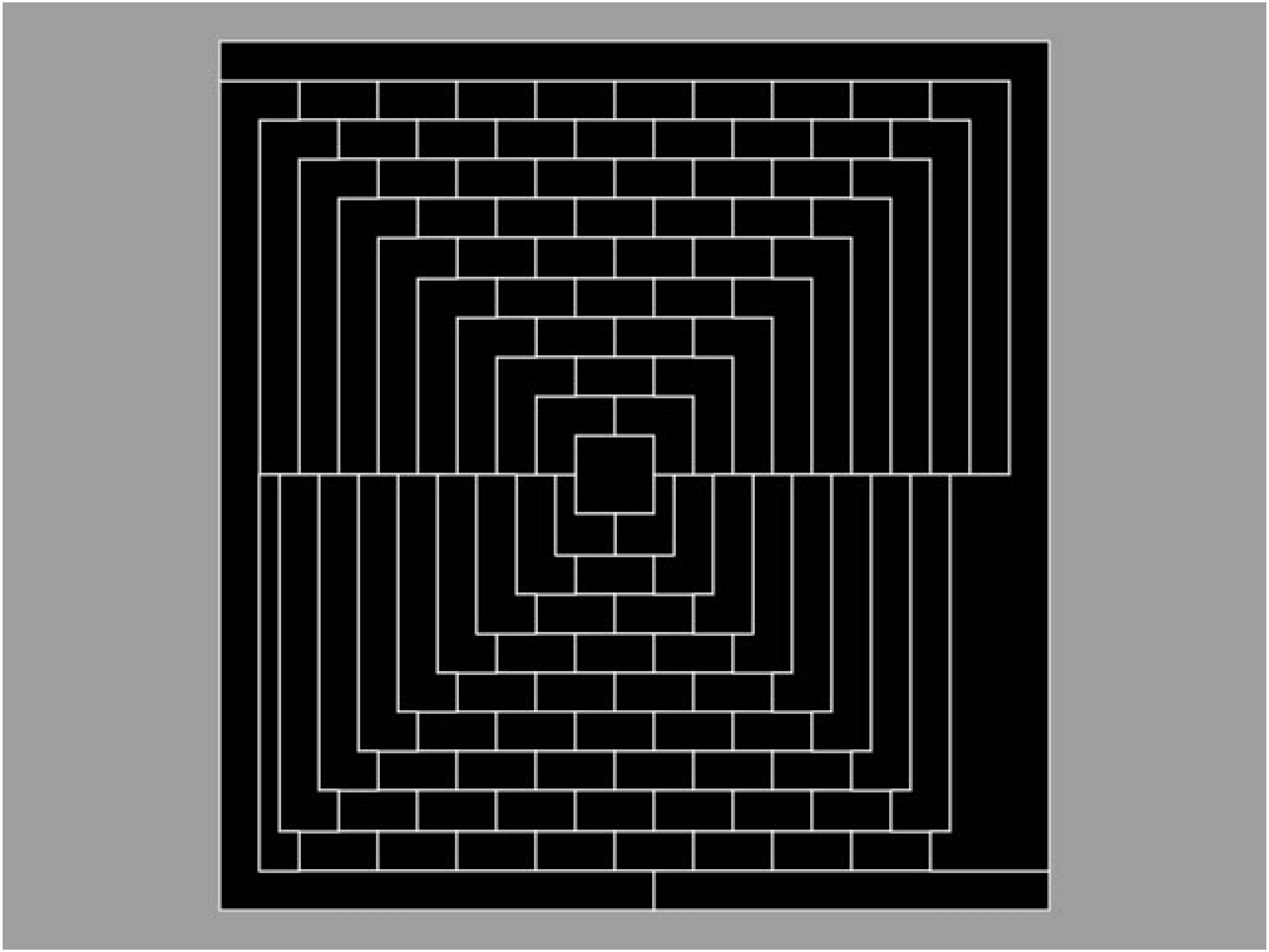
Based on backtracking search

- Try assigning values to variables
- When hit contradiction
  - Create new constraint encoding conflict
  - Backtrack by undoing some of the most recent assignments
  - Resume search with new variable assignments

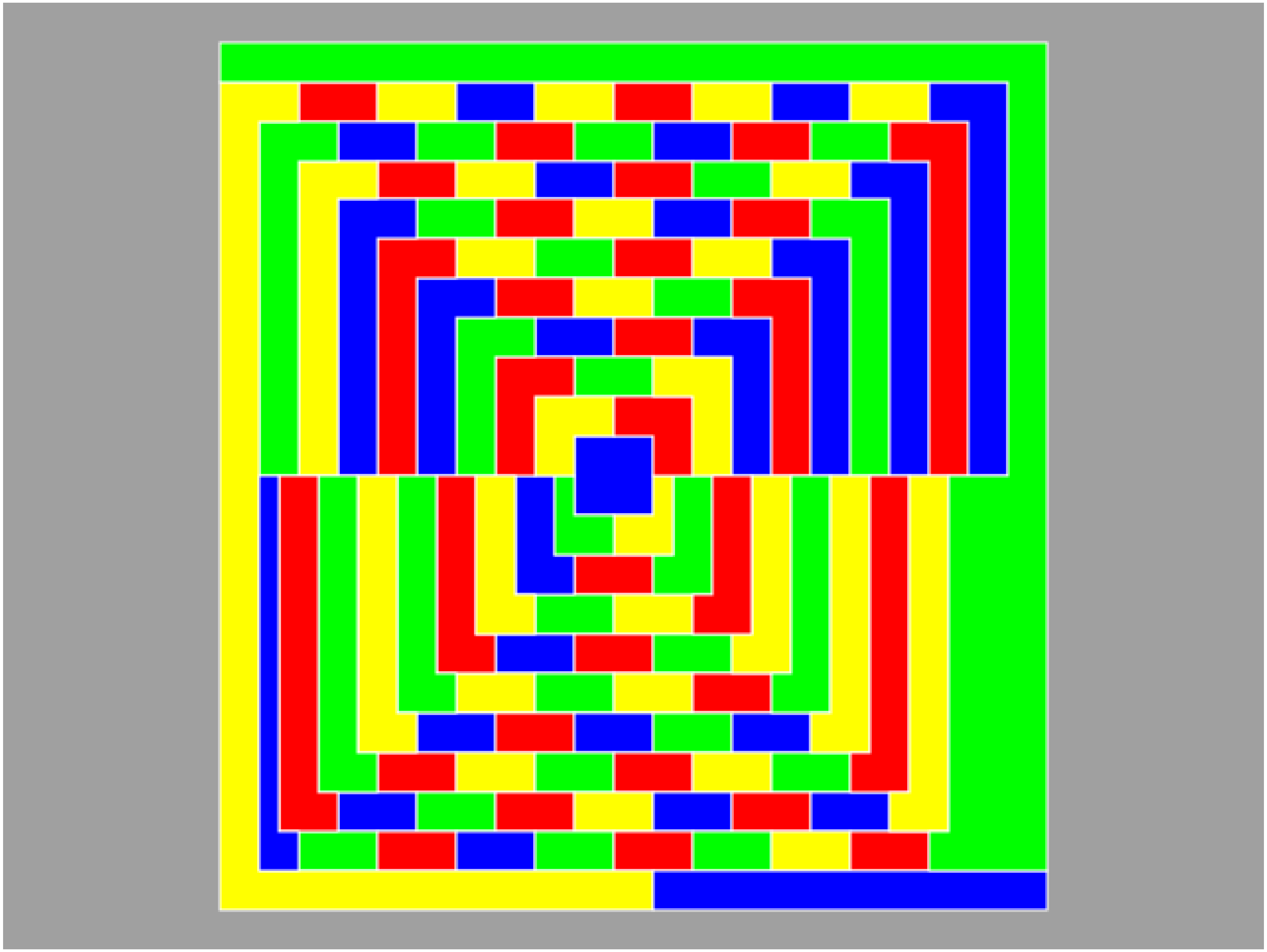
Visualizing the Search Process



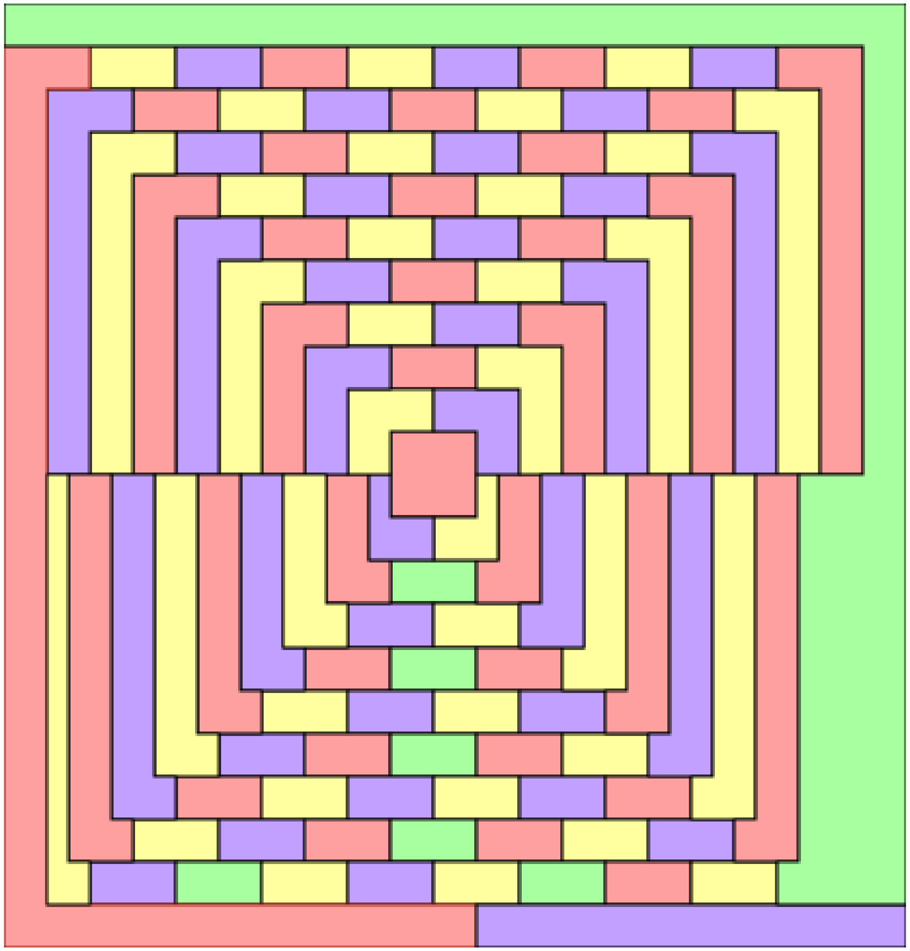
- **Black:** Neither variable assigned value
- **Single color:** Both variables assigned, giving unique color.
- **Blended colors:** One variable assigned, the other unassigned, indicating two possible colors
- **YouTube:** http://www.youtube.com/watch?v=0gt503wK7AI



The Final Result

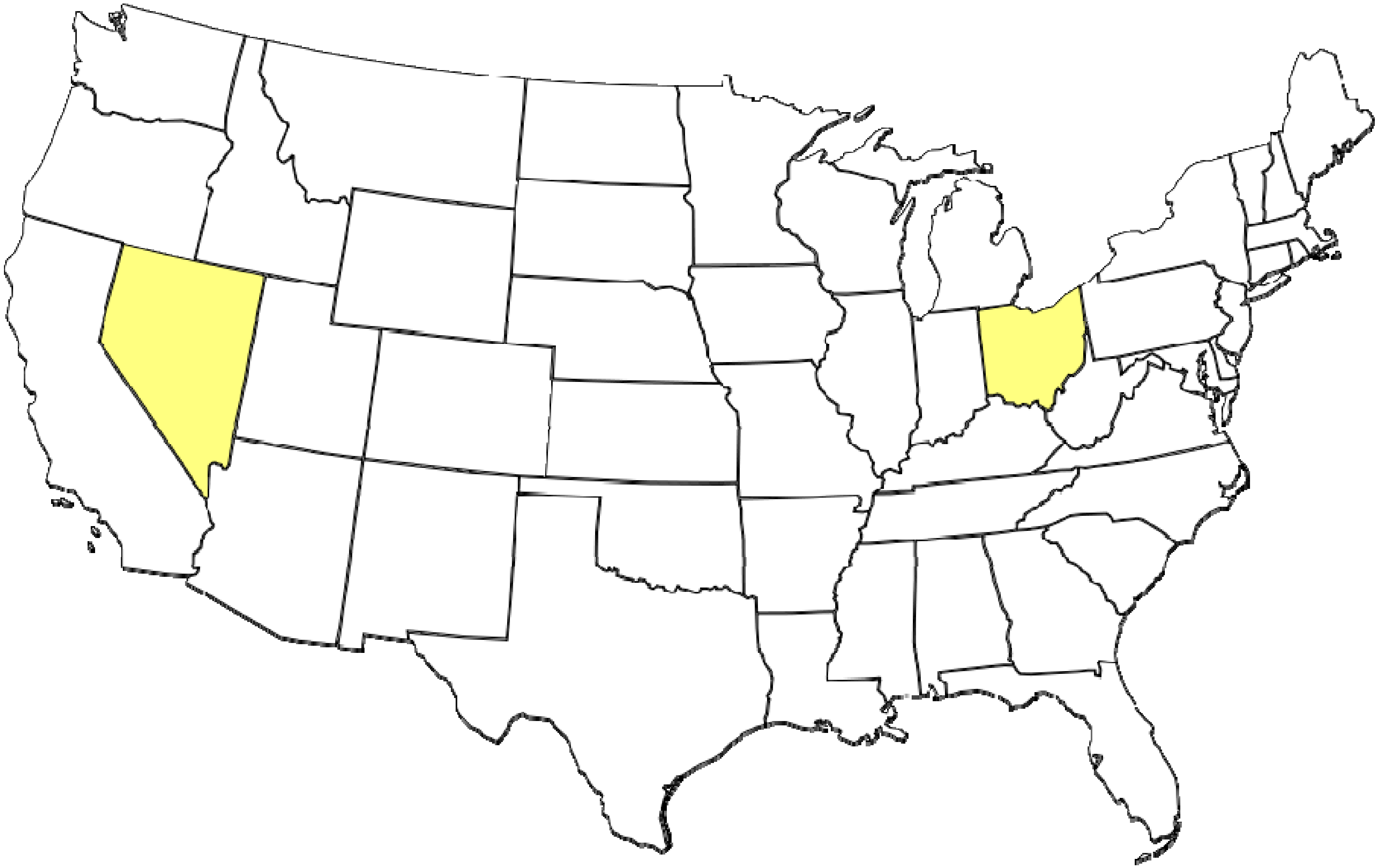


Another Solution



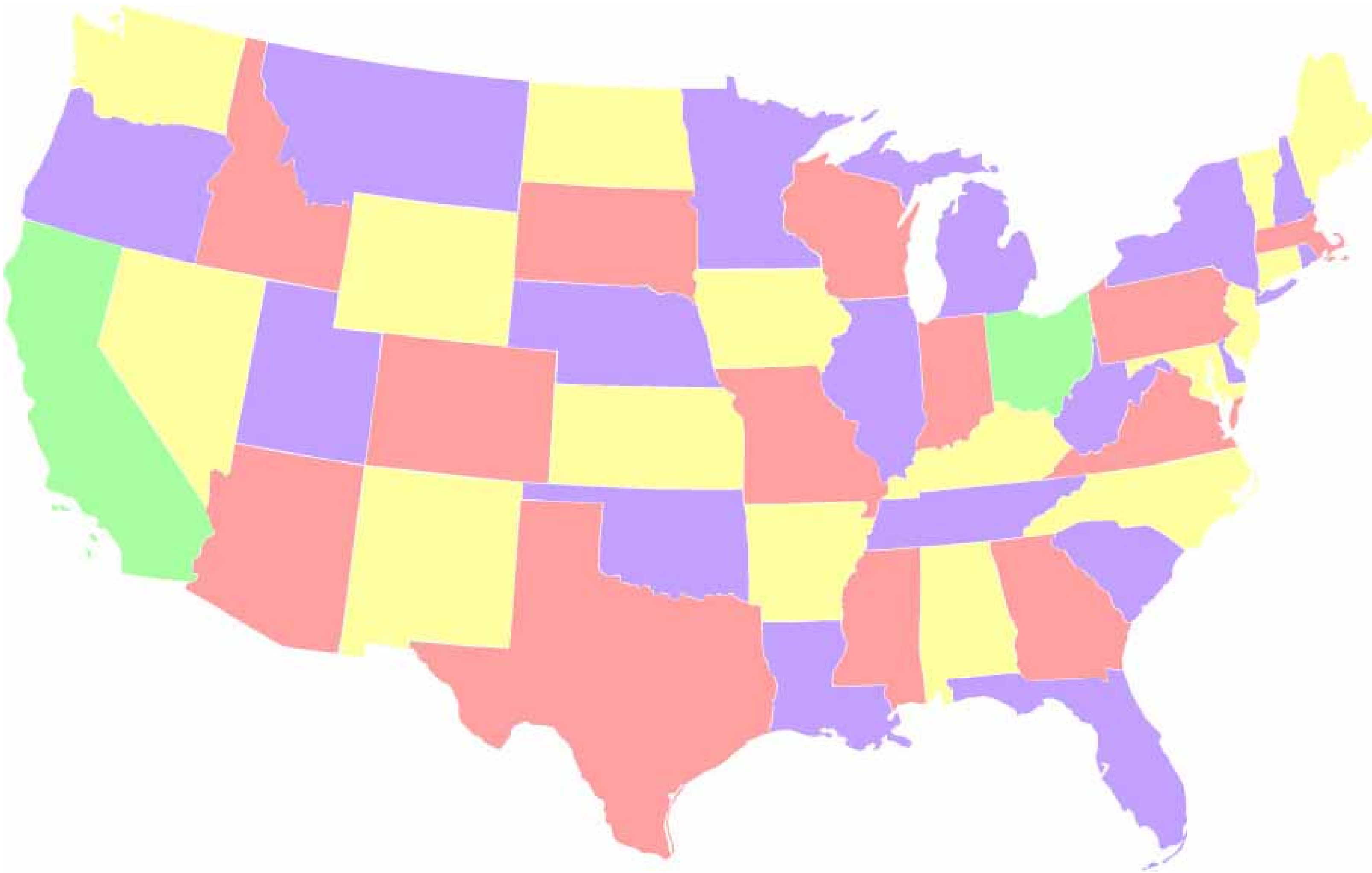
- Minimum use of greeen (7 times)

Try It Yourself



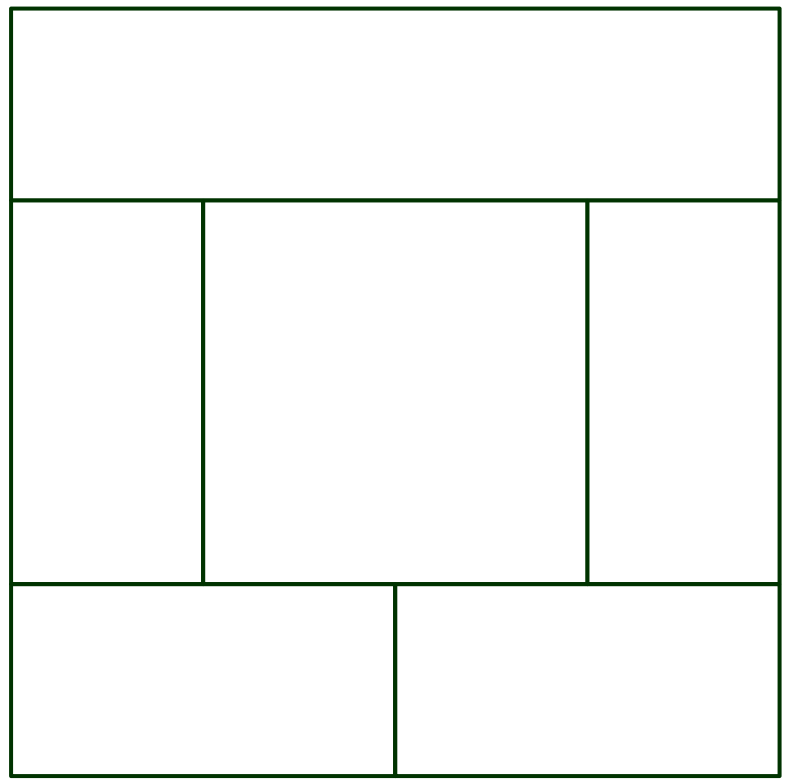
Color the rest of the map using 3 colors

Minimum Colorings of US Map



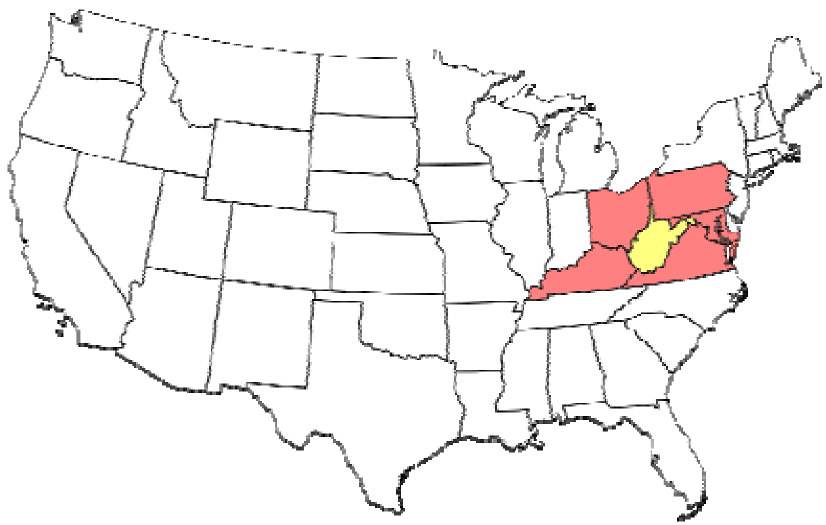
Only need to use green twice

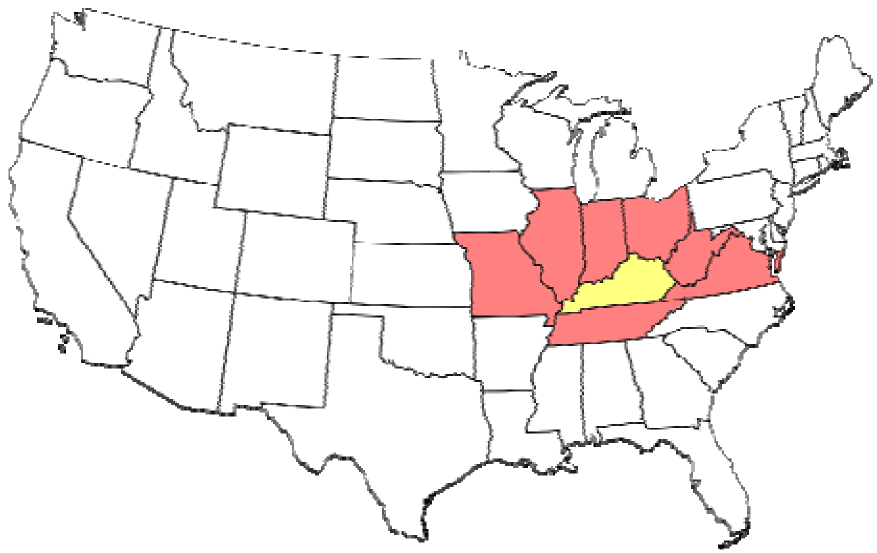
Odd Cycles

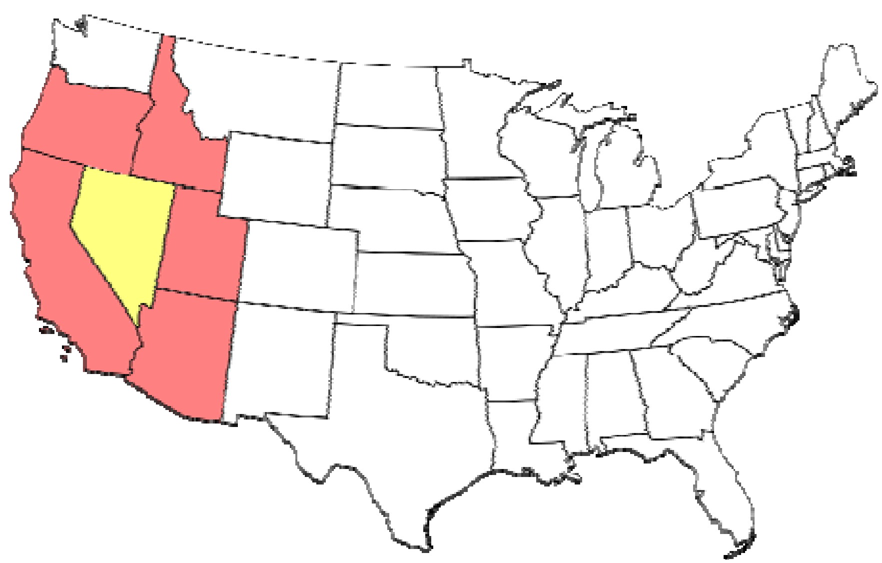


Can this be colored with
just 3 colors?

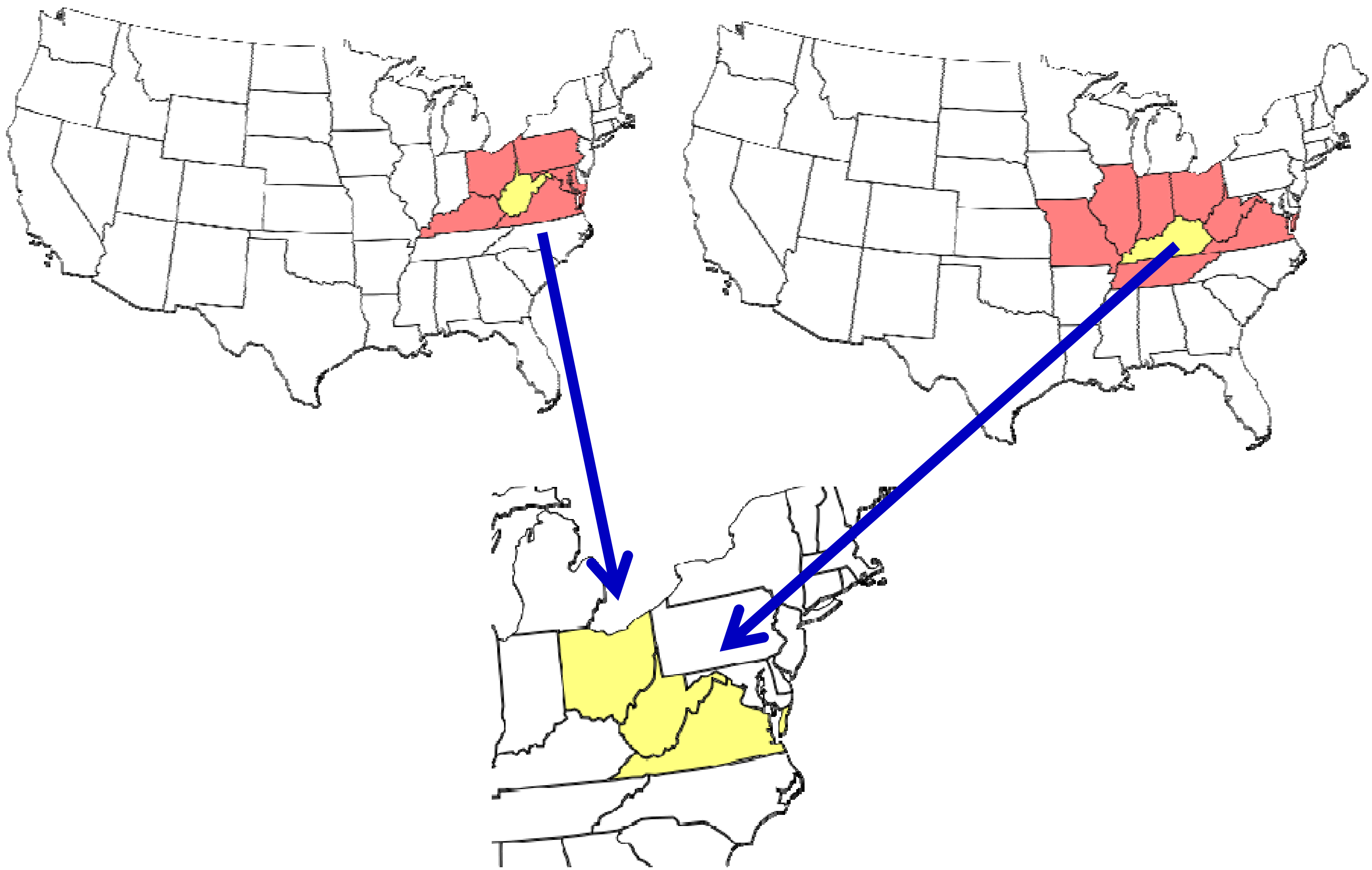
Odd Cycles in US Map



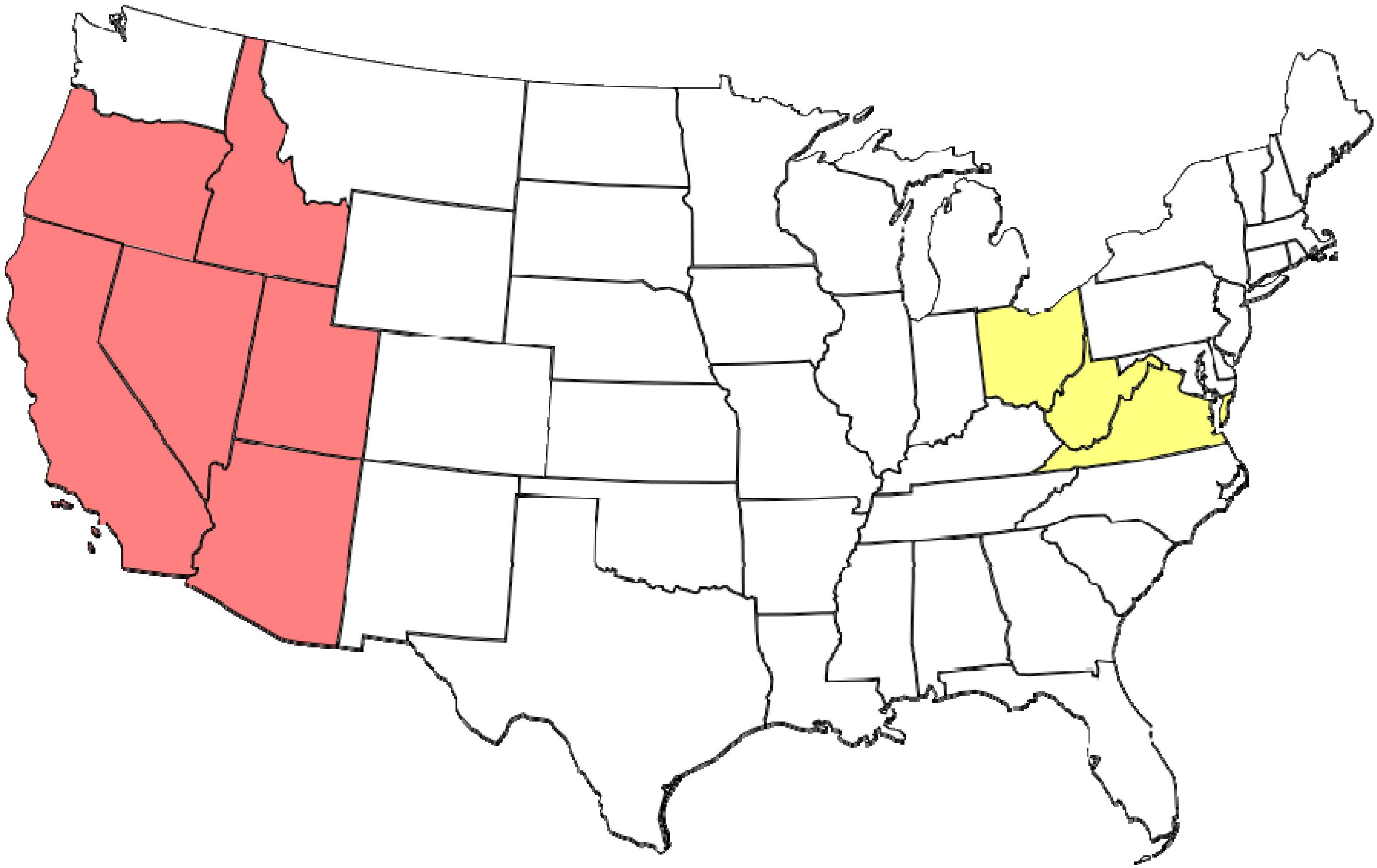




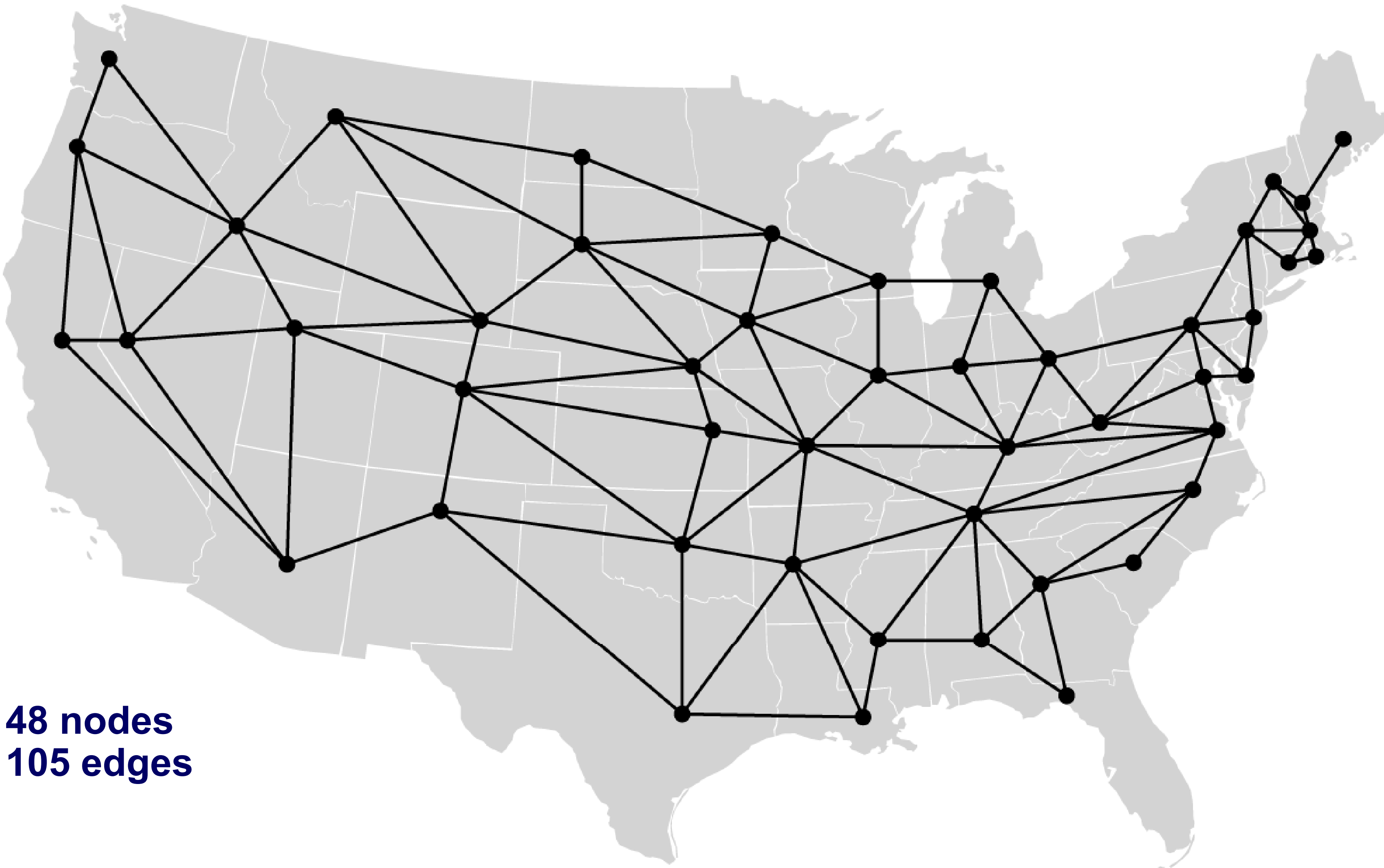
Overlapping Odd Cycles



Breaking Odd Cycles

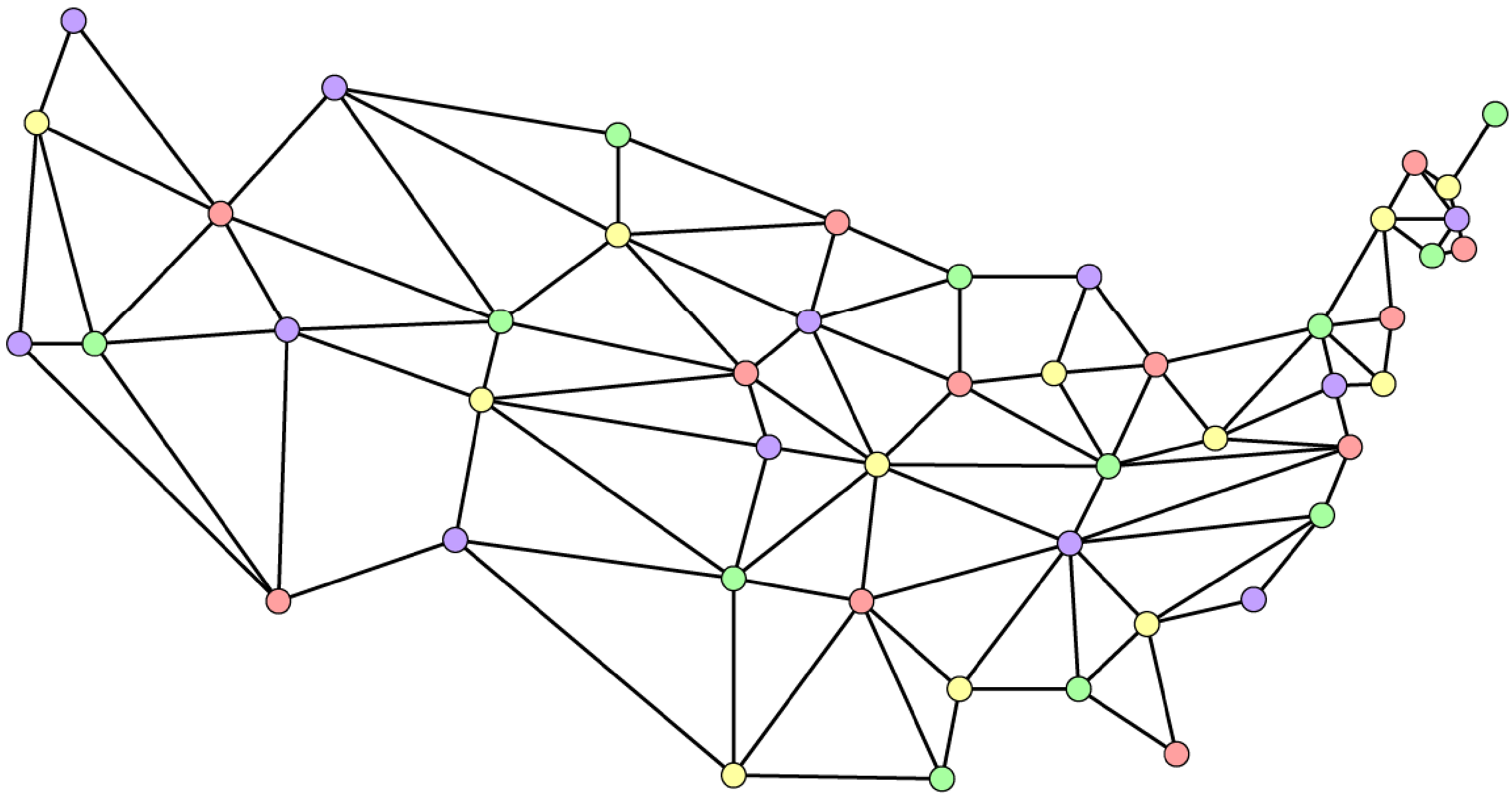


Viewing Maps as Graphs

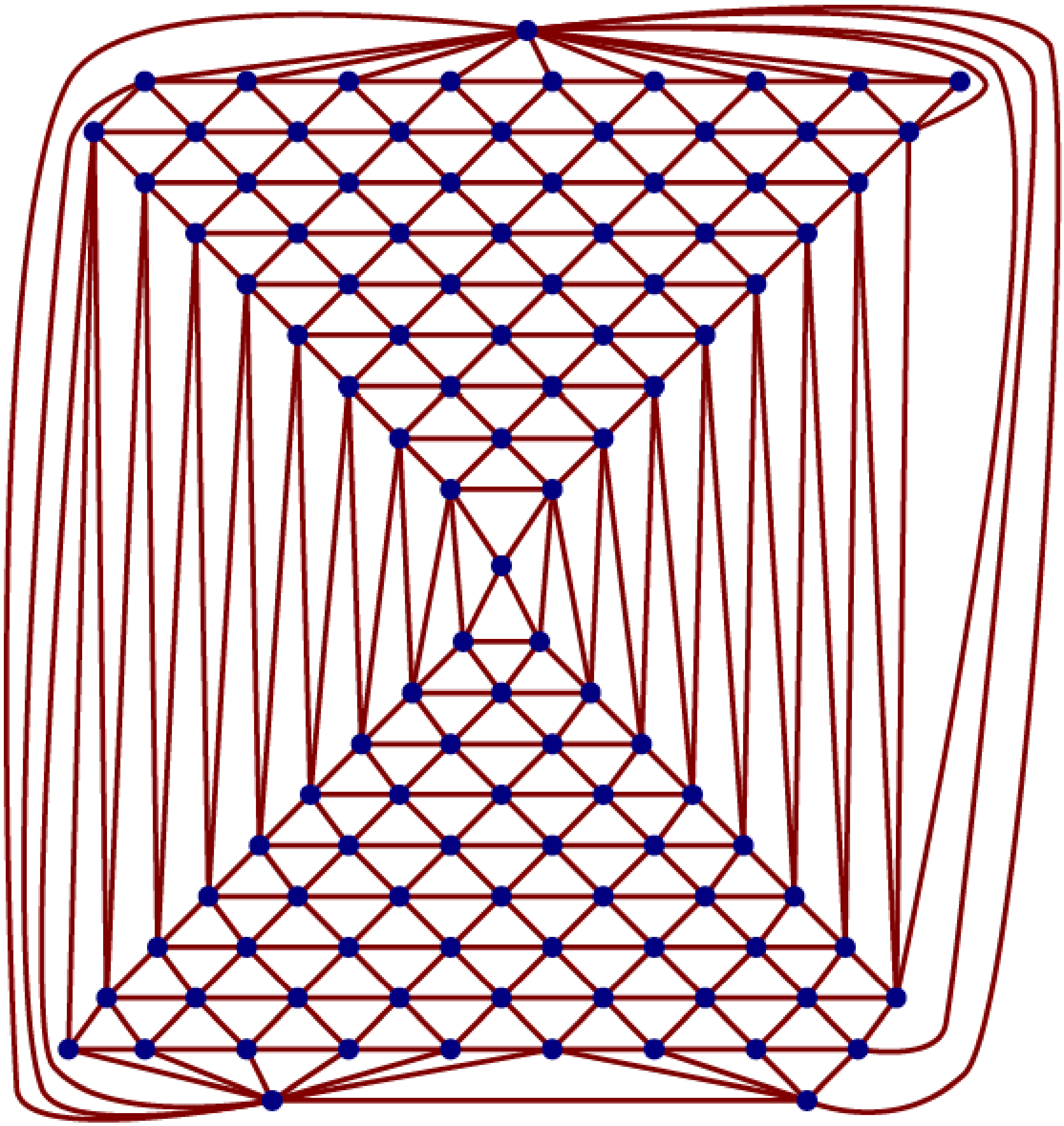


48 nodes
105 edges

Coloring a Graph



The Macgregor Graph



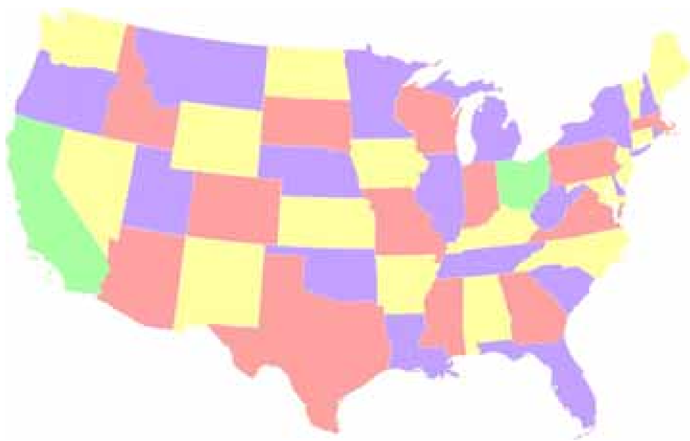
110 nodes
324 edges

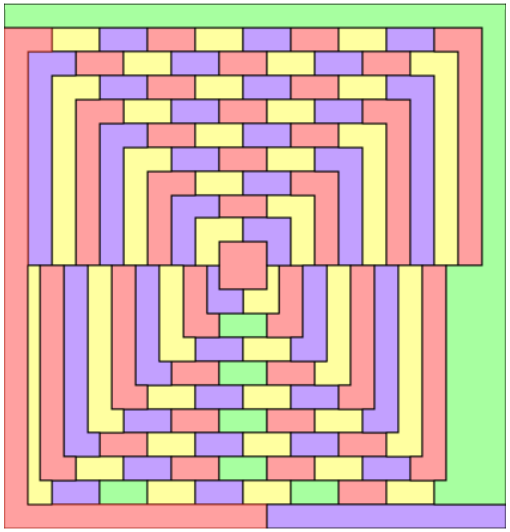
The Four Color Theorem

- Can color any planar graph with just 4 colors.

History

- Conjectured in 1852
- 1890: it was shown that 5 colors would suffice
- 1976: Appel & Haken claimed they had proof





Proof of Four Color Theorem

Proof Method

- Appel & Haaken showed there were 1,936 graphs that covered all possibilities
- Wrote computer program to check all of them

Reaction

- Many mathematicians didn't like this kind of proof
- Program has been rewritten and rechecked multiple times, and so the proof is generally accepted.

Coloring Other Graph Types

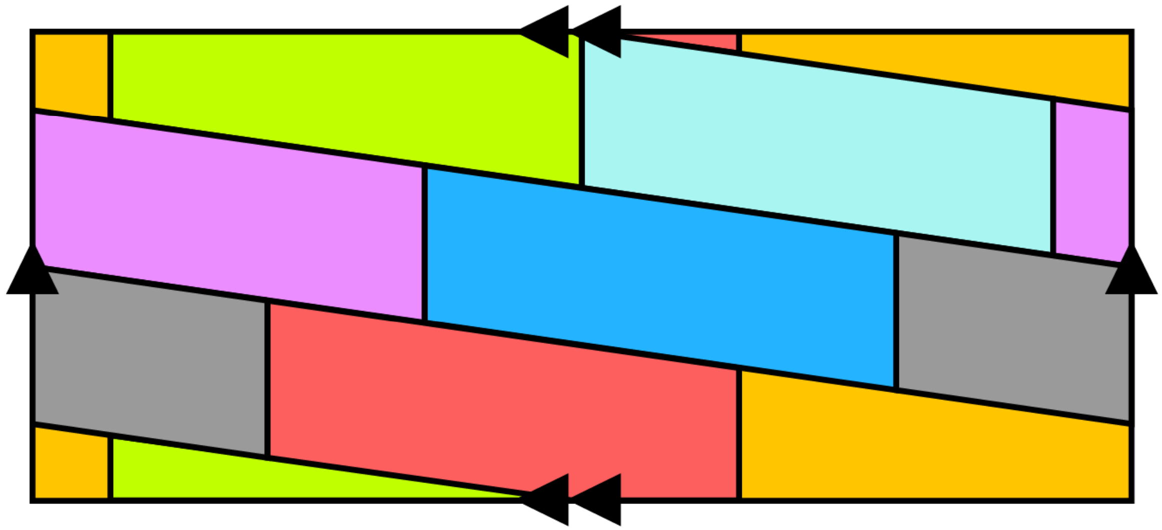
Sphere: same as plane

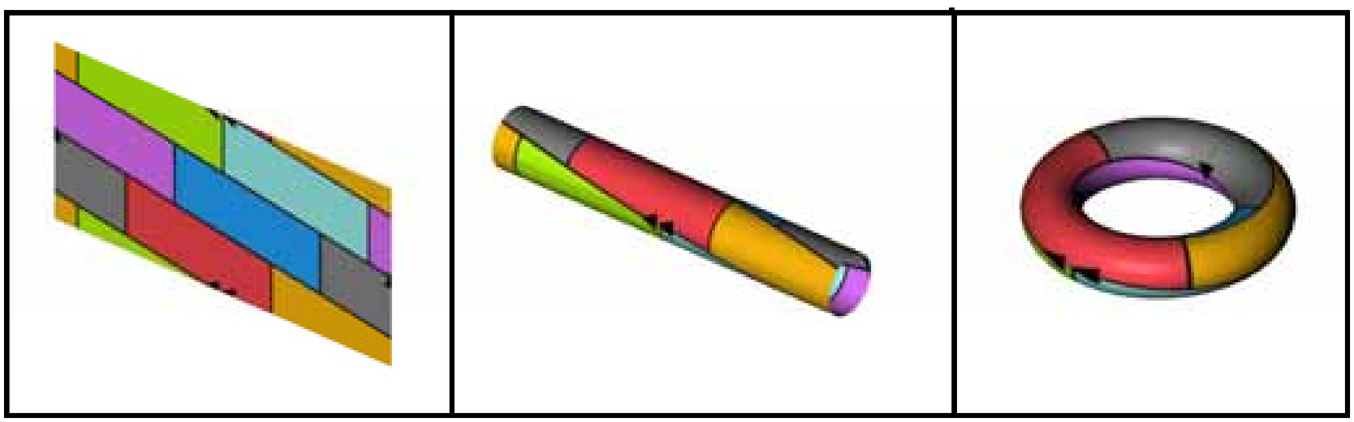
- **Plane → sphere**
  - Reduce exterior edges to points
- **Sphere → plane**
  - Cut hole and stretch out flat

Coloring A Torus

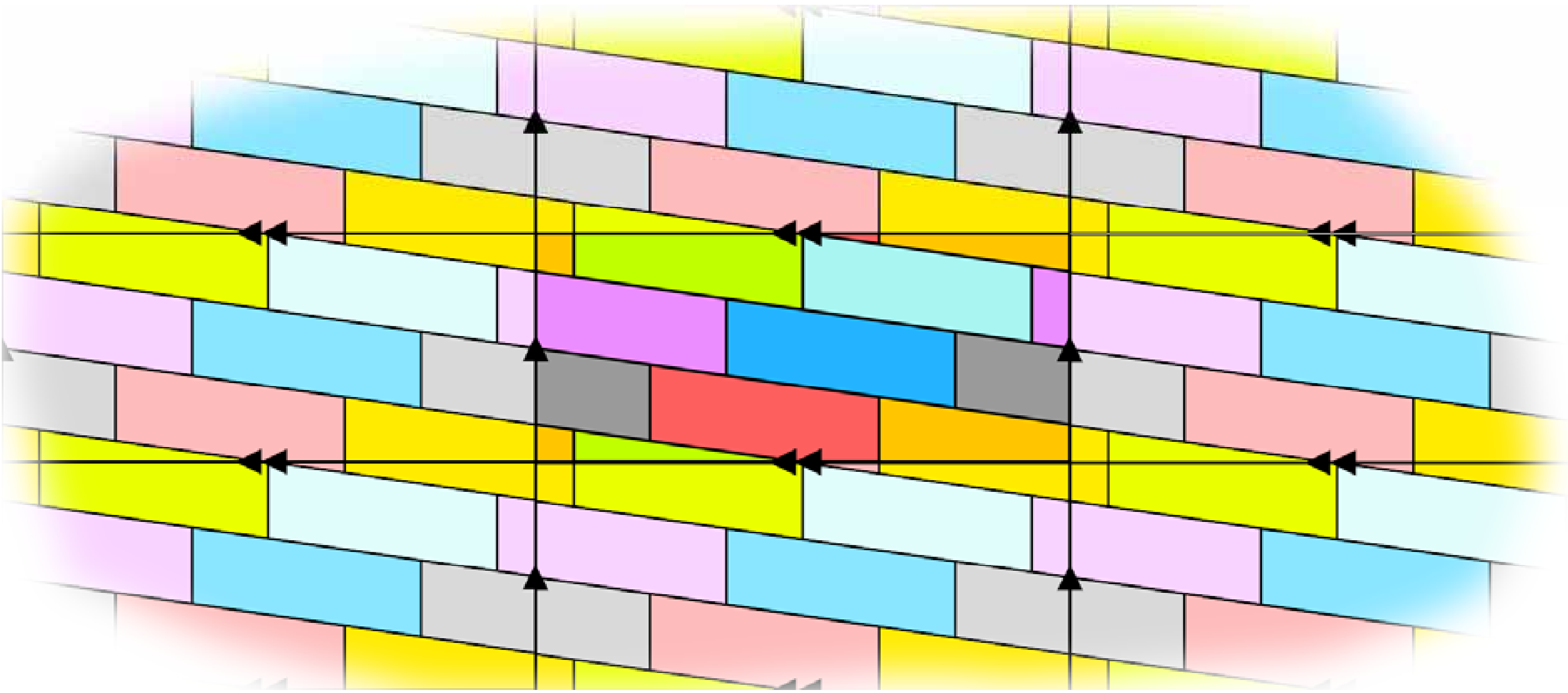
Torus

- **7 colors necessary**
  - 7 regions, each with 6 neighbors
- **Also sufficient**





Torus: An Infinite Wallpaper Pattern



Sudoku as a Graph Coloring Problem

				4				
			2					
	4	2						
					2			
							2	

3	9	6	7	4	1	2	5	8
8	5	7	2	3	9	1	4	6
1	2	4	5	8	6	7	9	3
5	7	1	6	2	3	9	8	4
6	8	3	4	9	7	5	1	2
9	4	2	1	5	8	3	6	7
2	3	9	8	1	4	6	7	5
4	6	5	9	7	2	8	3	1
7	1	8	3	6	5	4	2	9

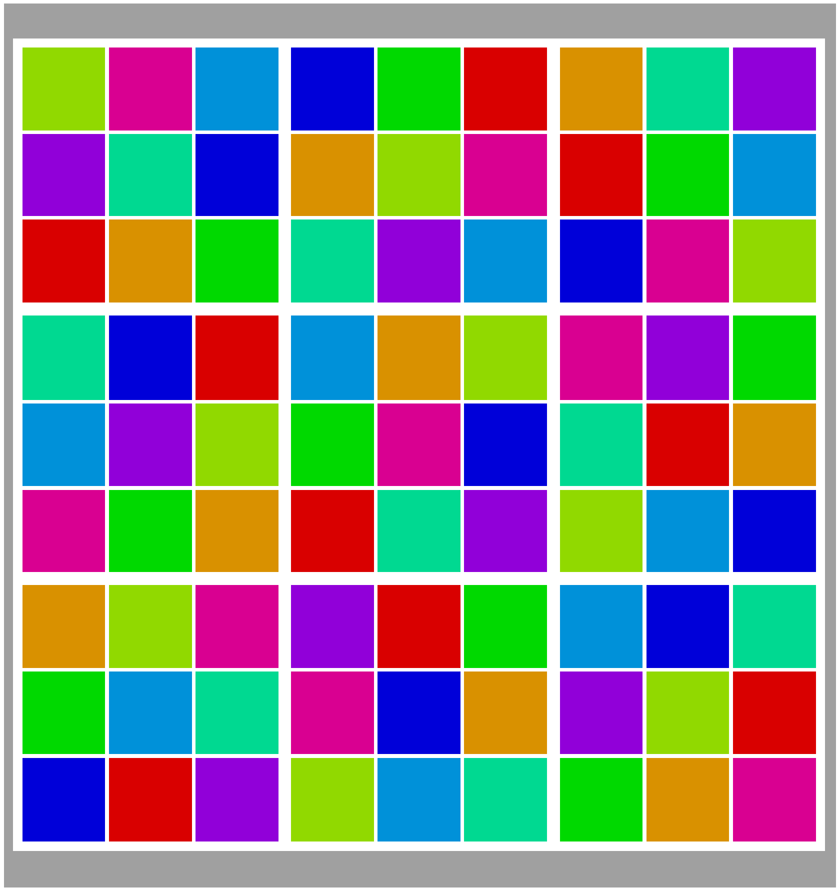
Adding Colors

1
2
3
4
5
6
7
8
9

3	9	6	7	4	1	2	5	8
8	5	7	2	3	9	1	4	6
1	2	4	5	8	6	7	9	3
5	7	1	6	2	3	9	8	4
6	8	3	4	9	7	5	1	2
9	4	2	1	5	8	3	6	7
2	3	9	8	1	4	6	7	5
4	6	5	9	7	2	8	3	1
7	1	8	3	6	5	4	2	9

Taking Away Numbers

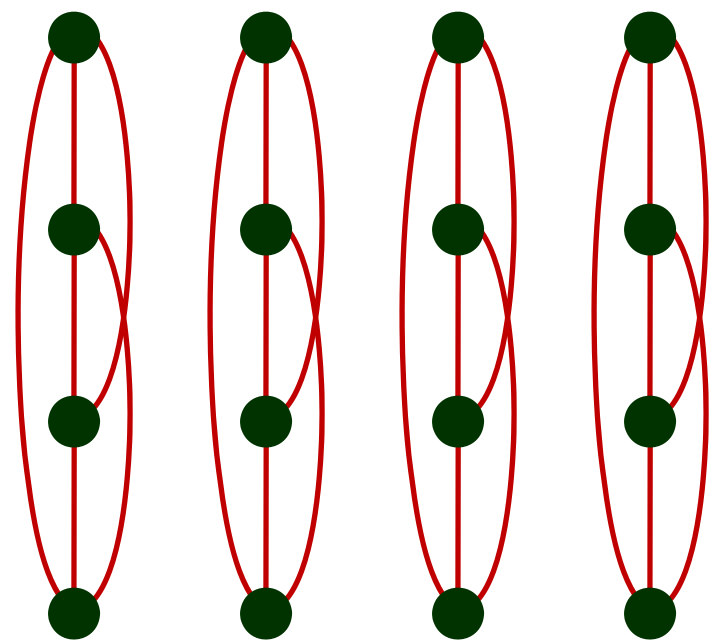
3	9	6	7	4	1	2	5	8
8	5	7	2	3	9	1	4	6
1	2	4	5	8	6	7	9	3
5	7	1	6	2	3	9	8	4
6	8	3	4	9	7	5	1	2
9	4	2	1	5	8	3	6	7
2	3	9	8	1	4	6	7	5
4	6	5	9	7	2	8	3	1
7	1	8	3	6	5	4	2	9



Graph Structure of Sudoku

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

4 X 4 Sudoku

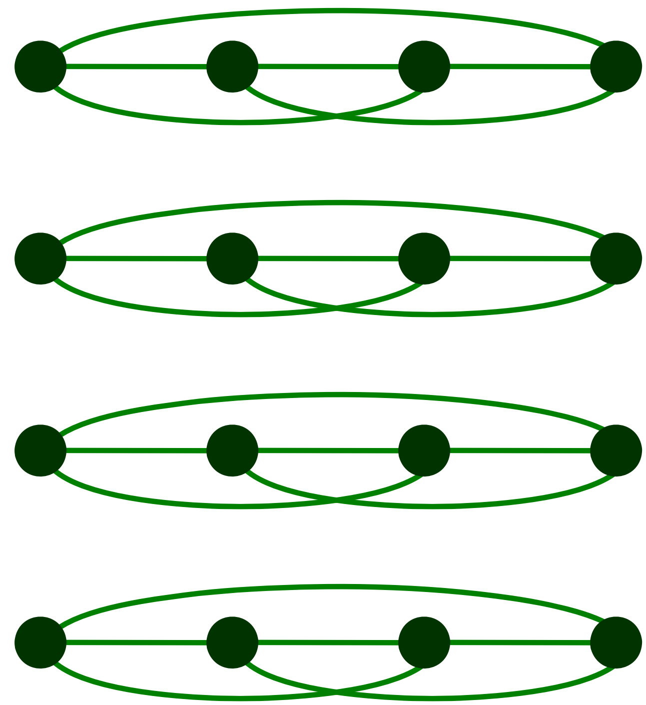


Column constraints

Graph Structure of Sudoku

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

4 X 4 Sudoku

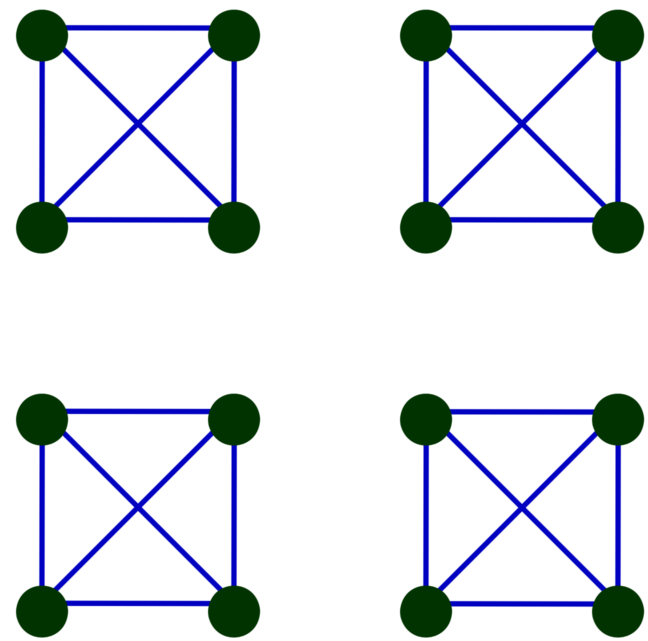


Row constraints

Graph Structure of Sudoku

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

4 X 4 Sudoku

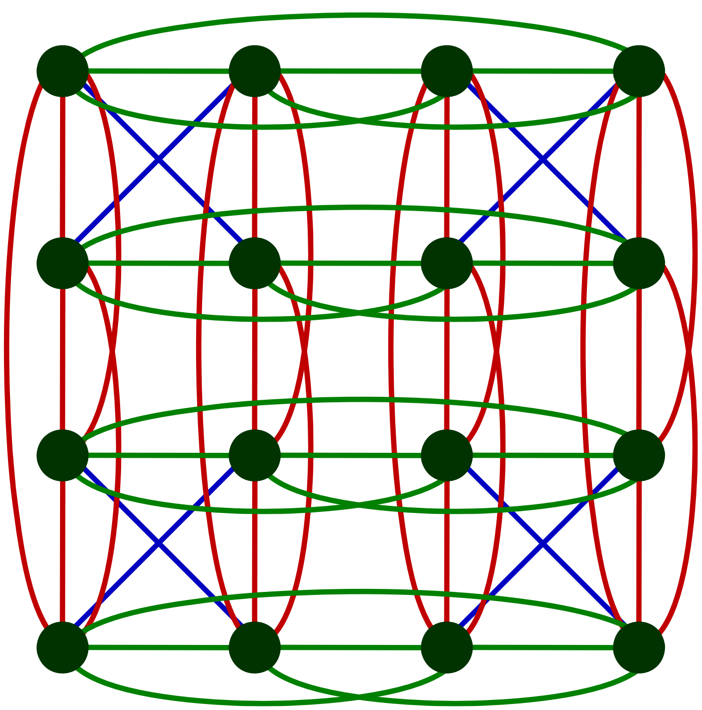


Block constraints

Graph Structure of Sudoku

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

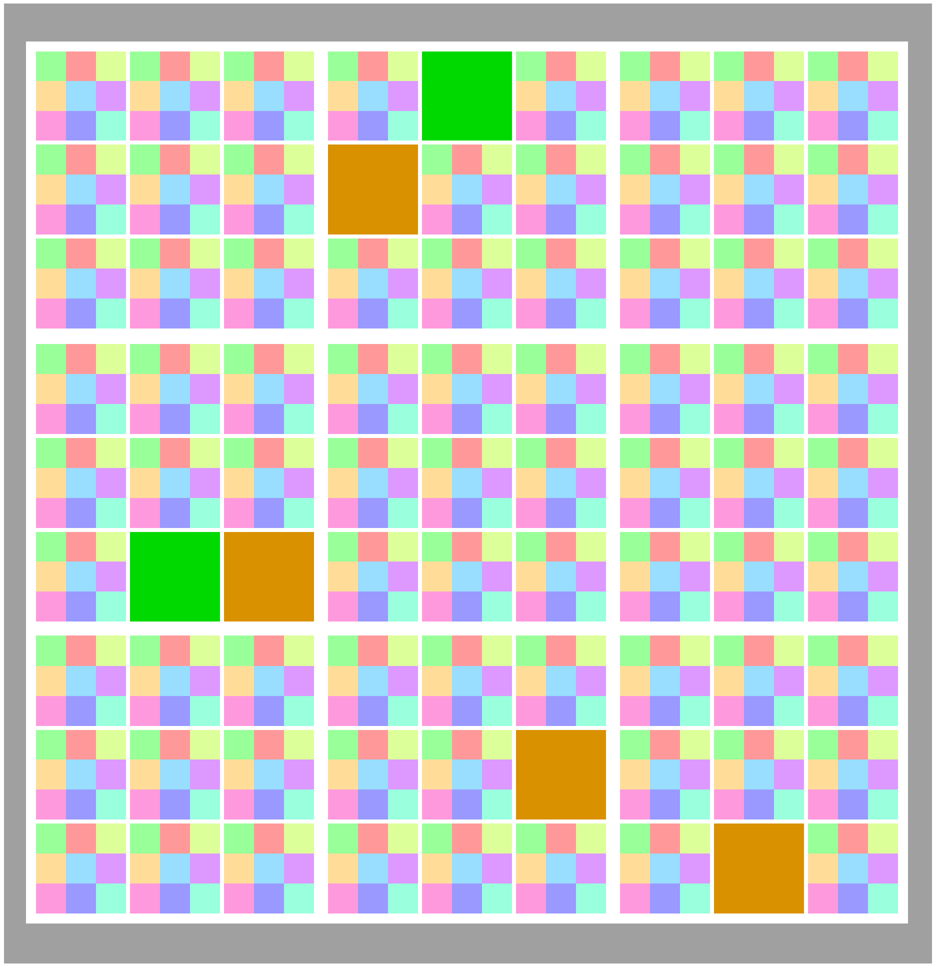
4 X 4 Sudoku
16 nodes

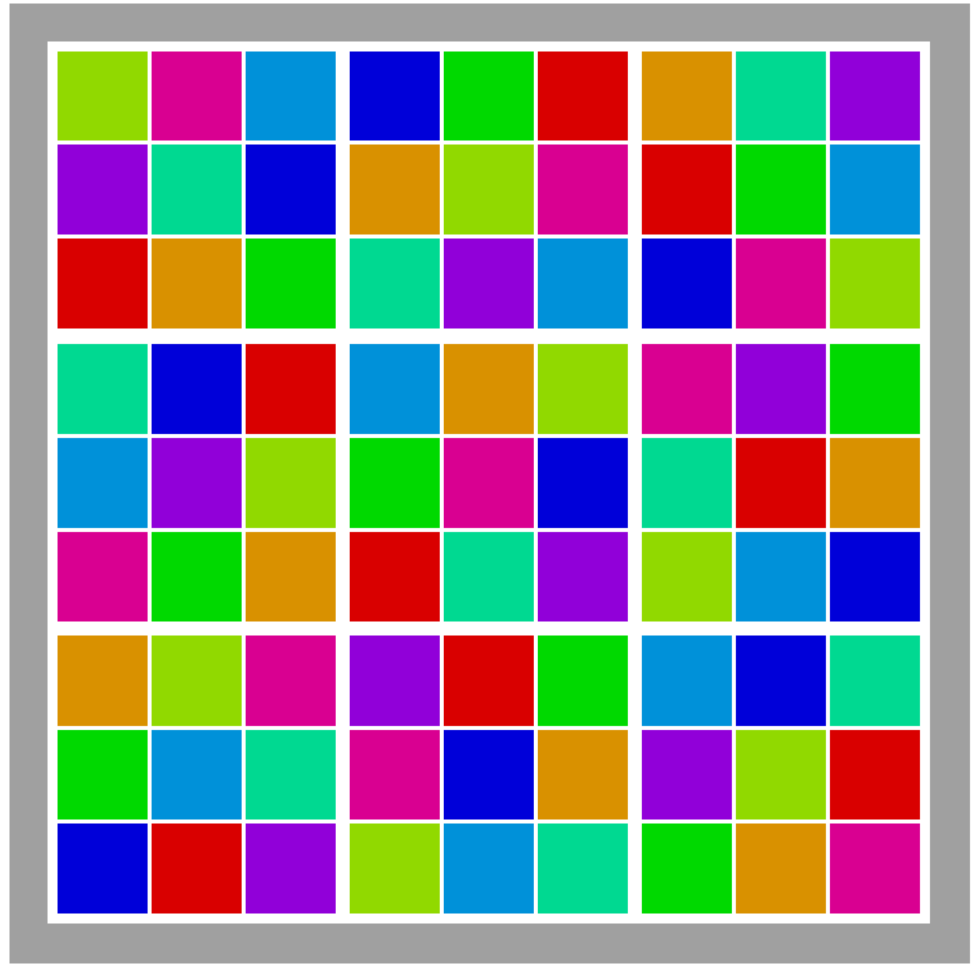


All constraints
56 edges

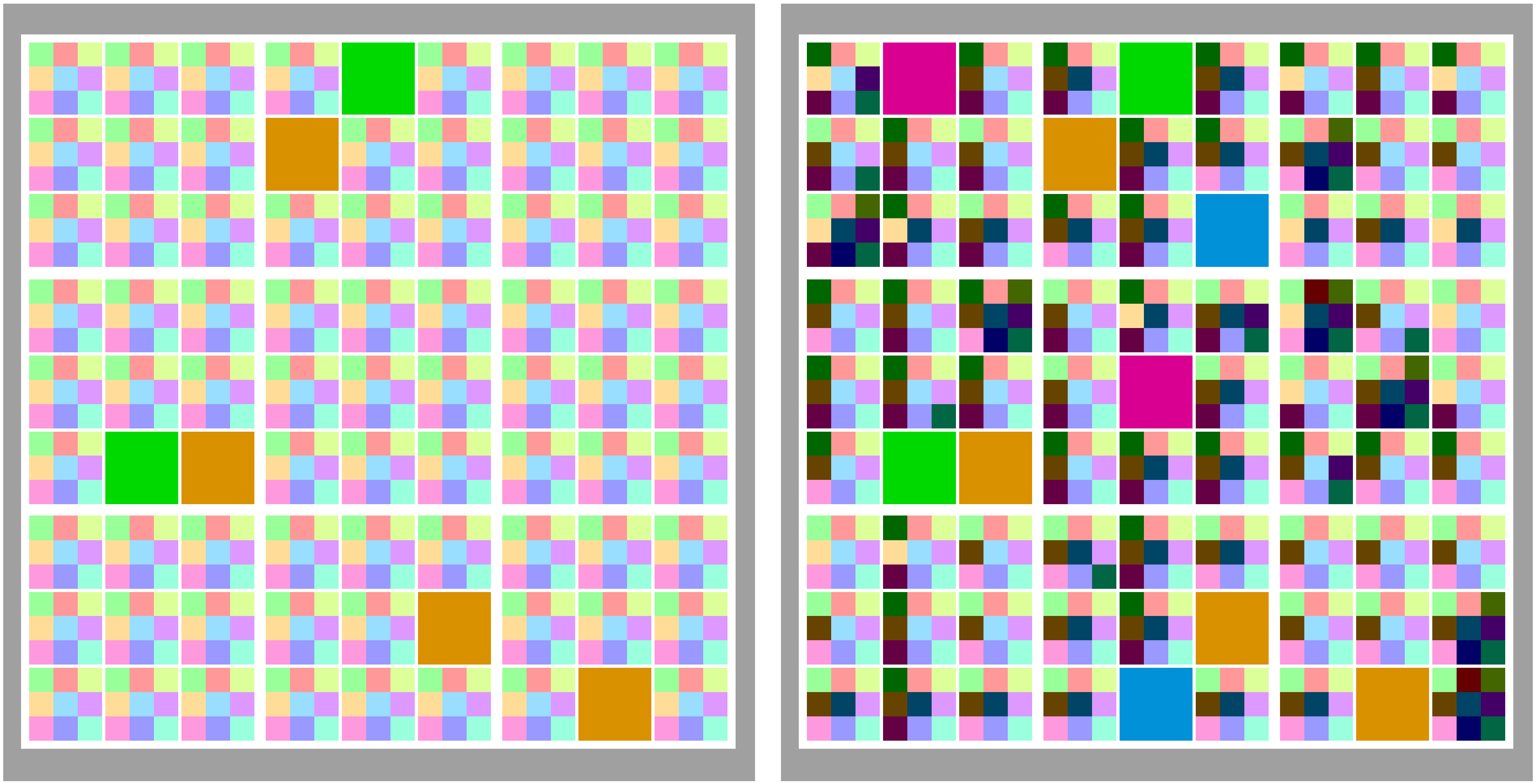
9 x 9 Sudoku: 81 nodes, 810 edges

Visualizing Solution Process

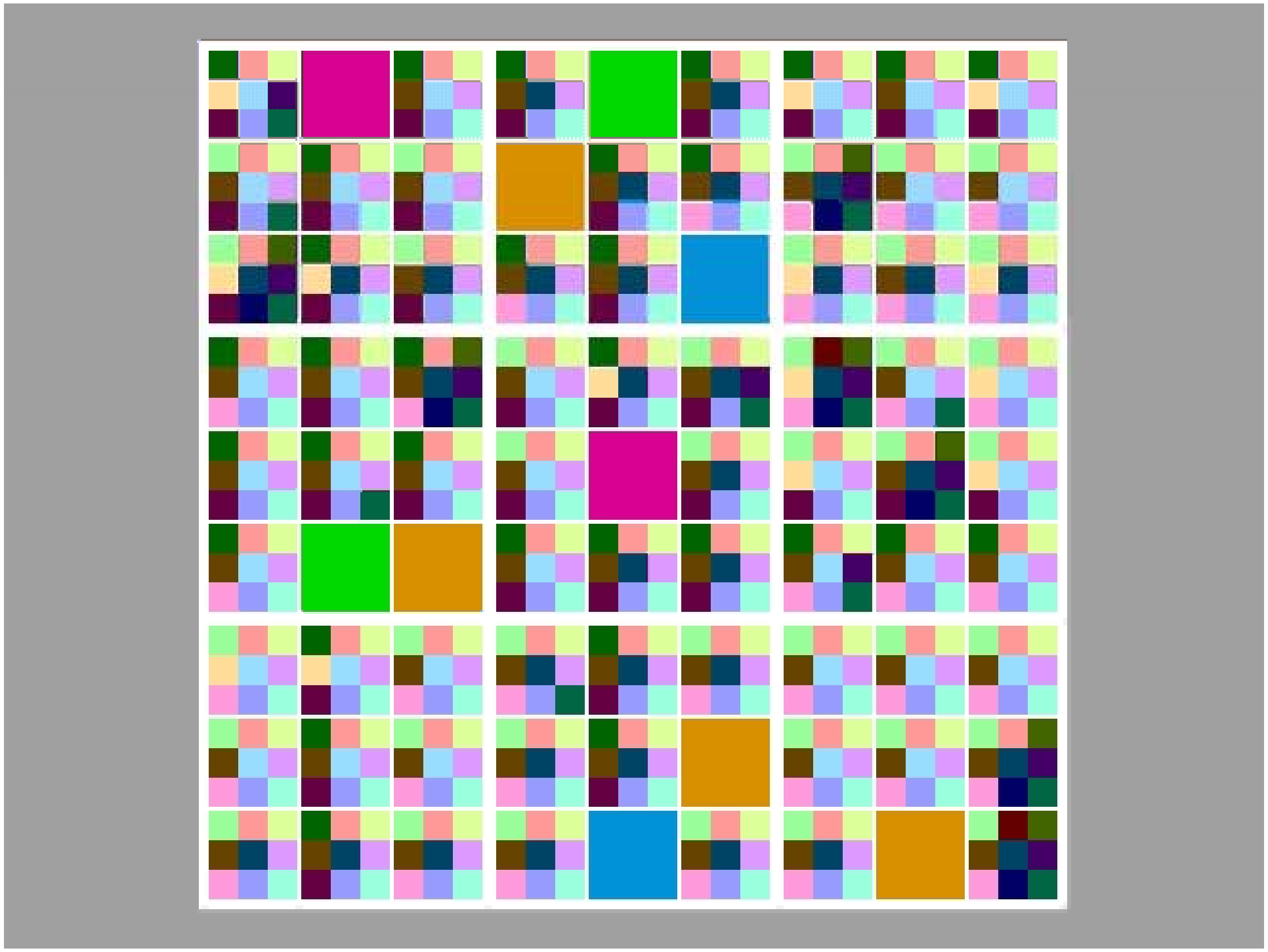




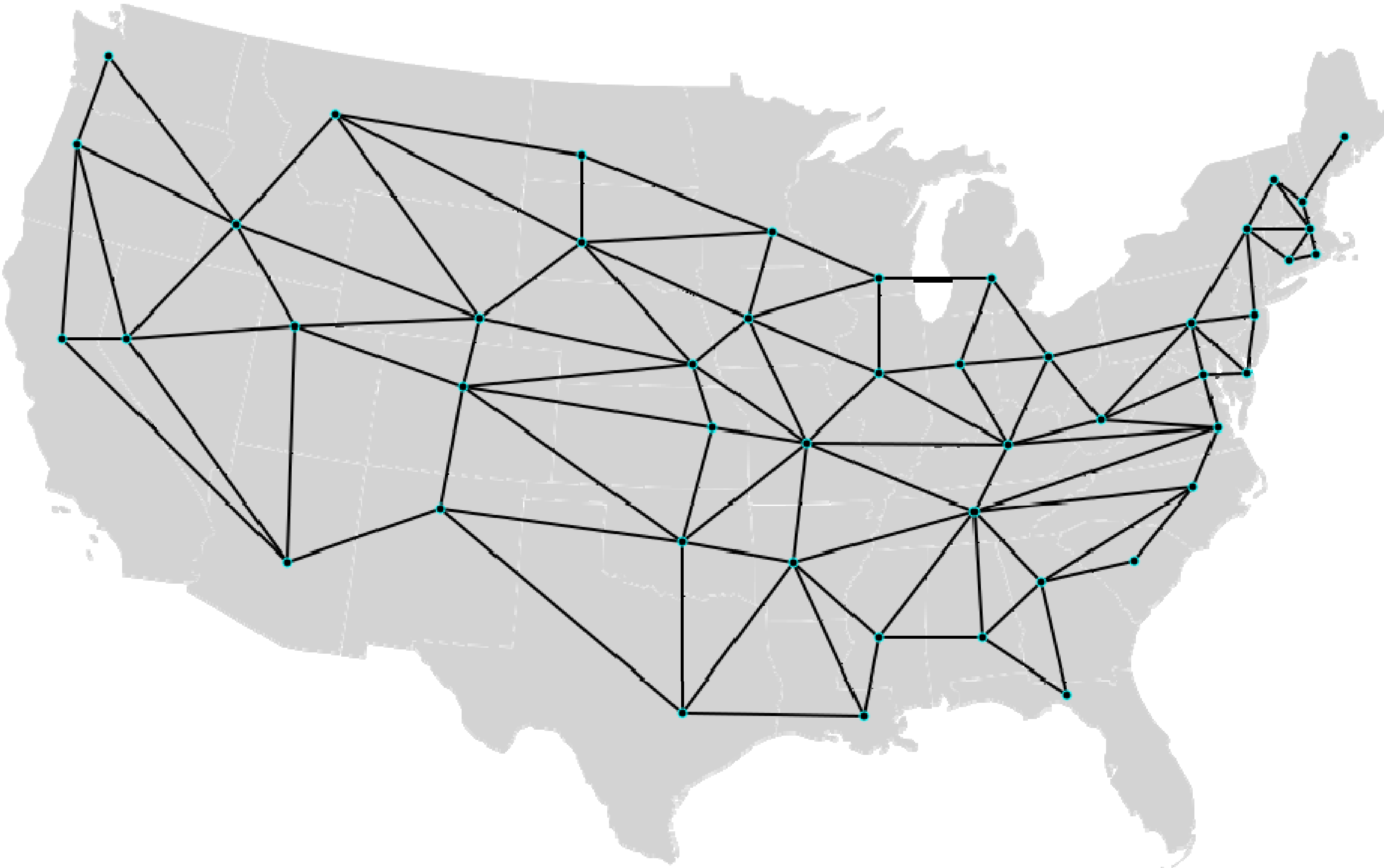
Visualizing Solution Process



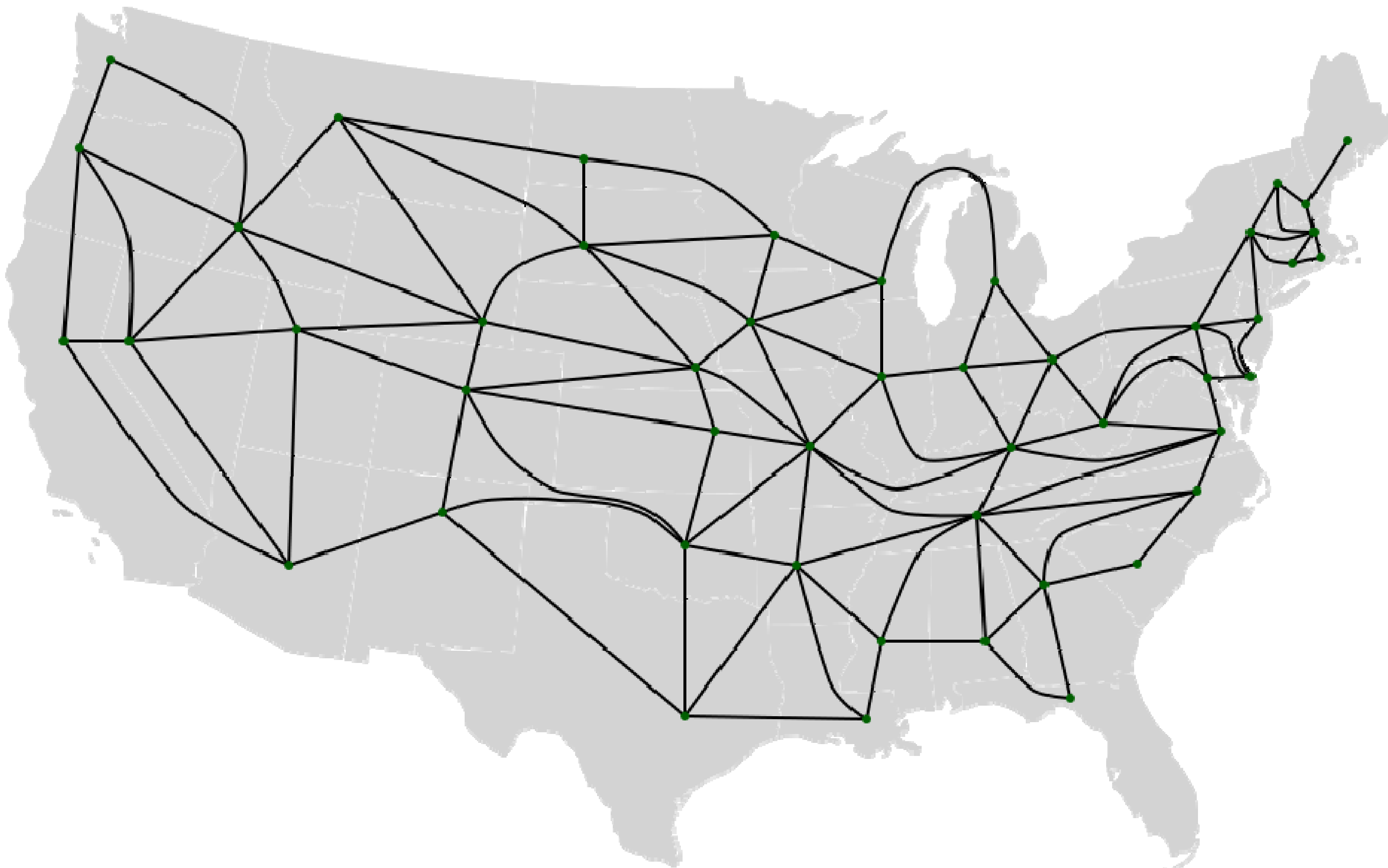
Solving A Sudoku Puzzle



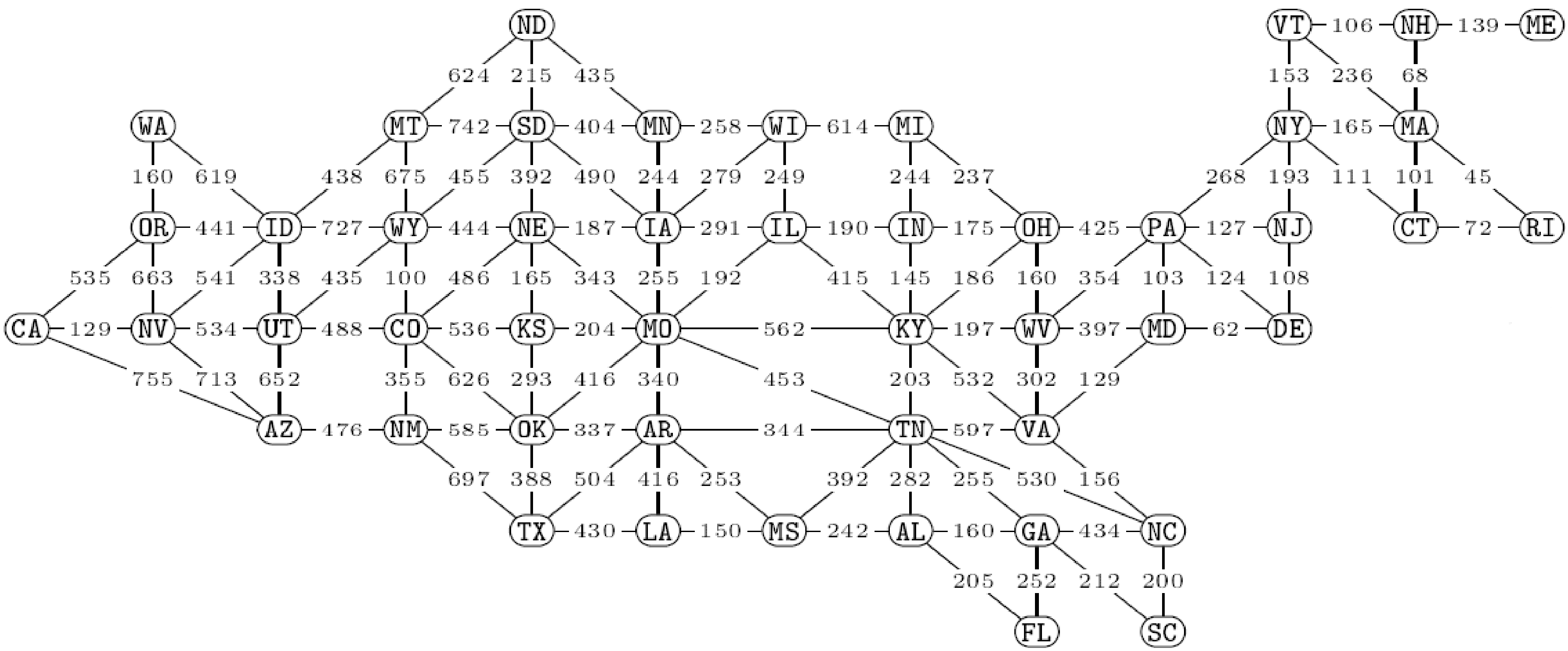
Touring the US



Touring the US

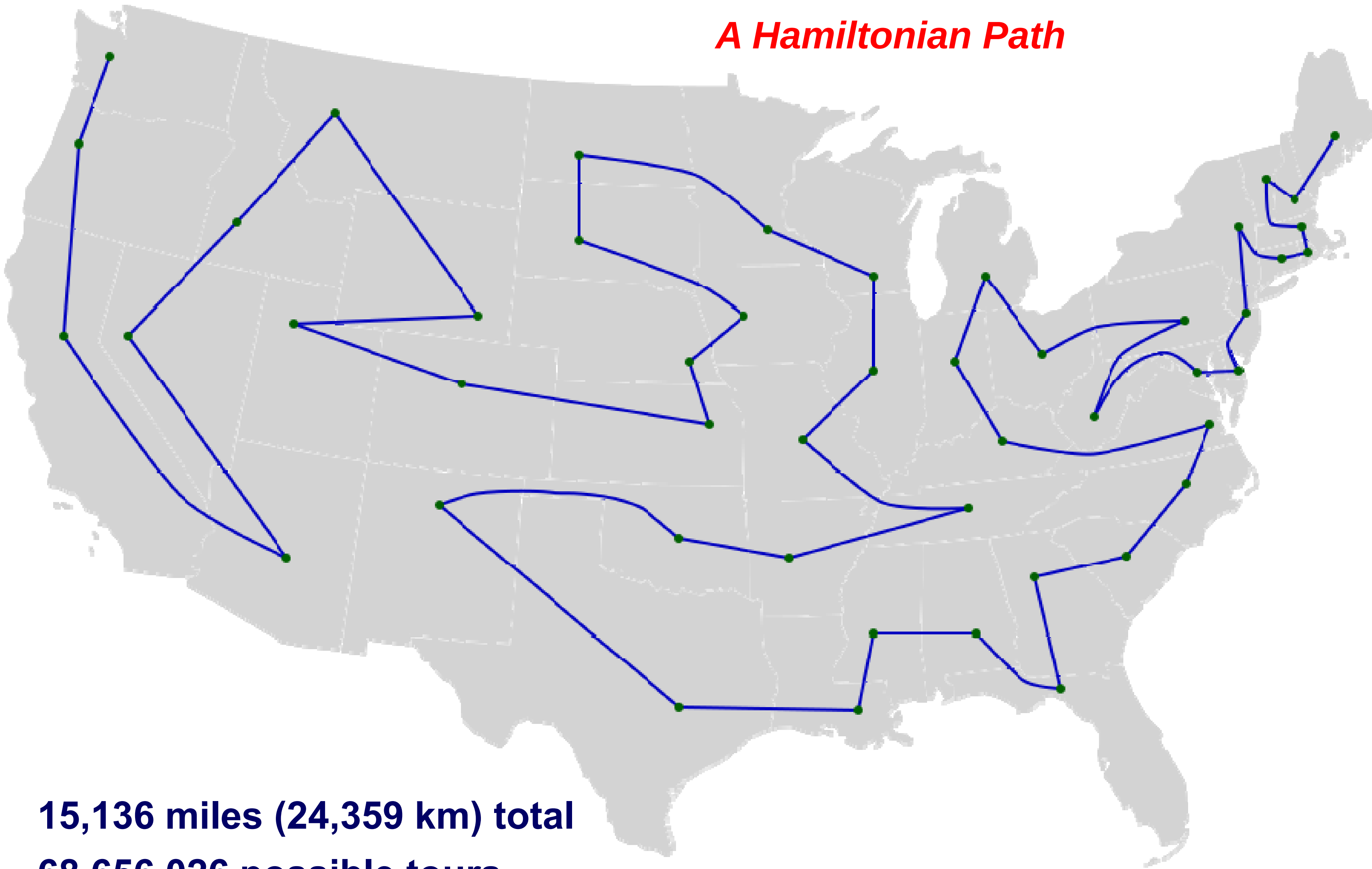


Weighted US Graph



- **Shortest driving distances between capitol cities**
  - Staying within source and destination states
  - Computed by Don Knuth using Mapquest

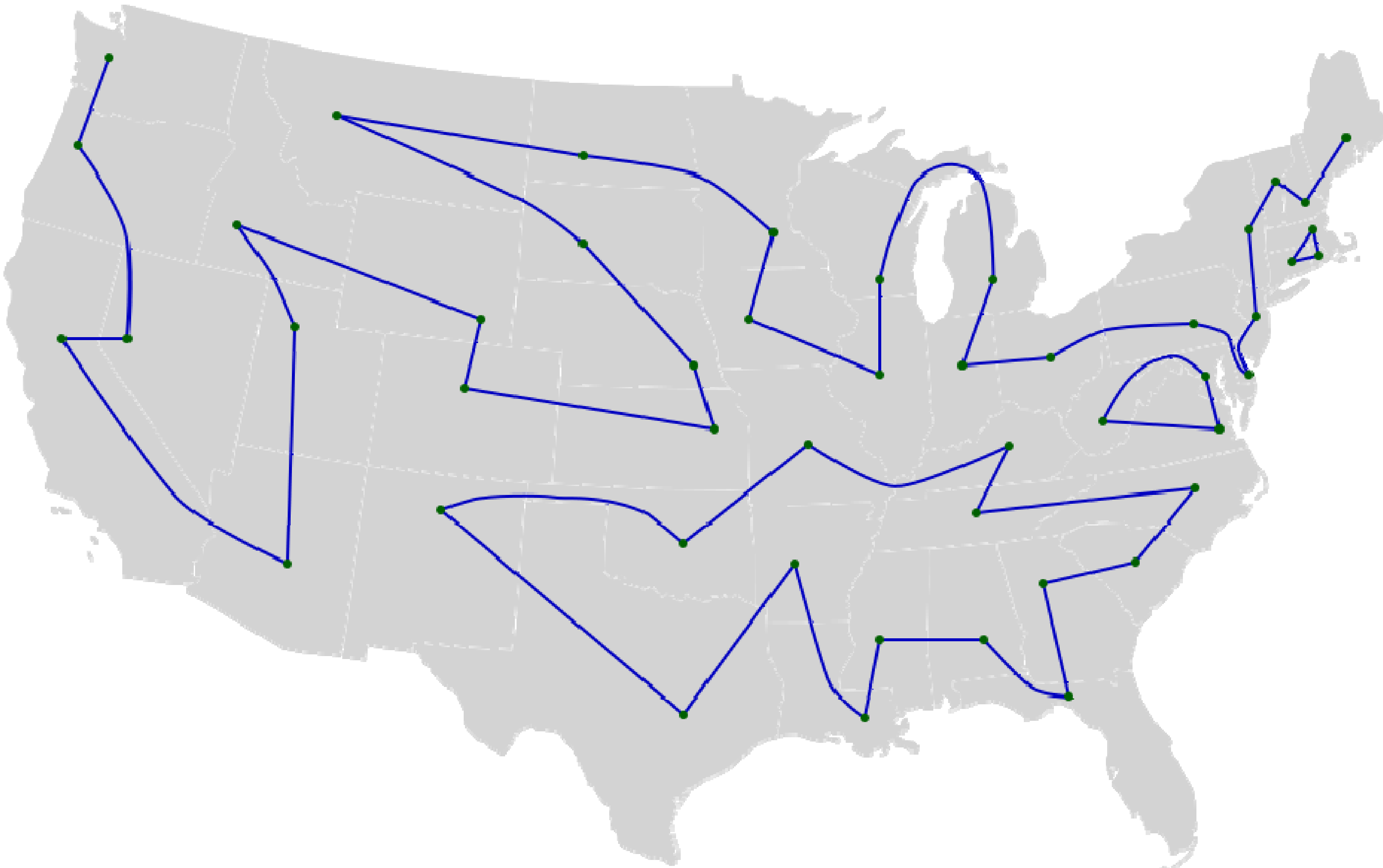
A Capitol Tour



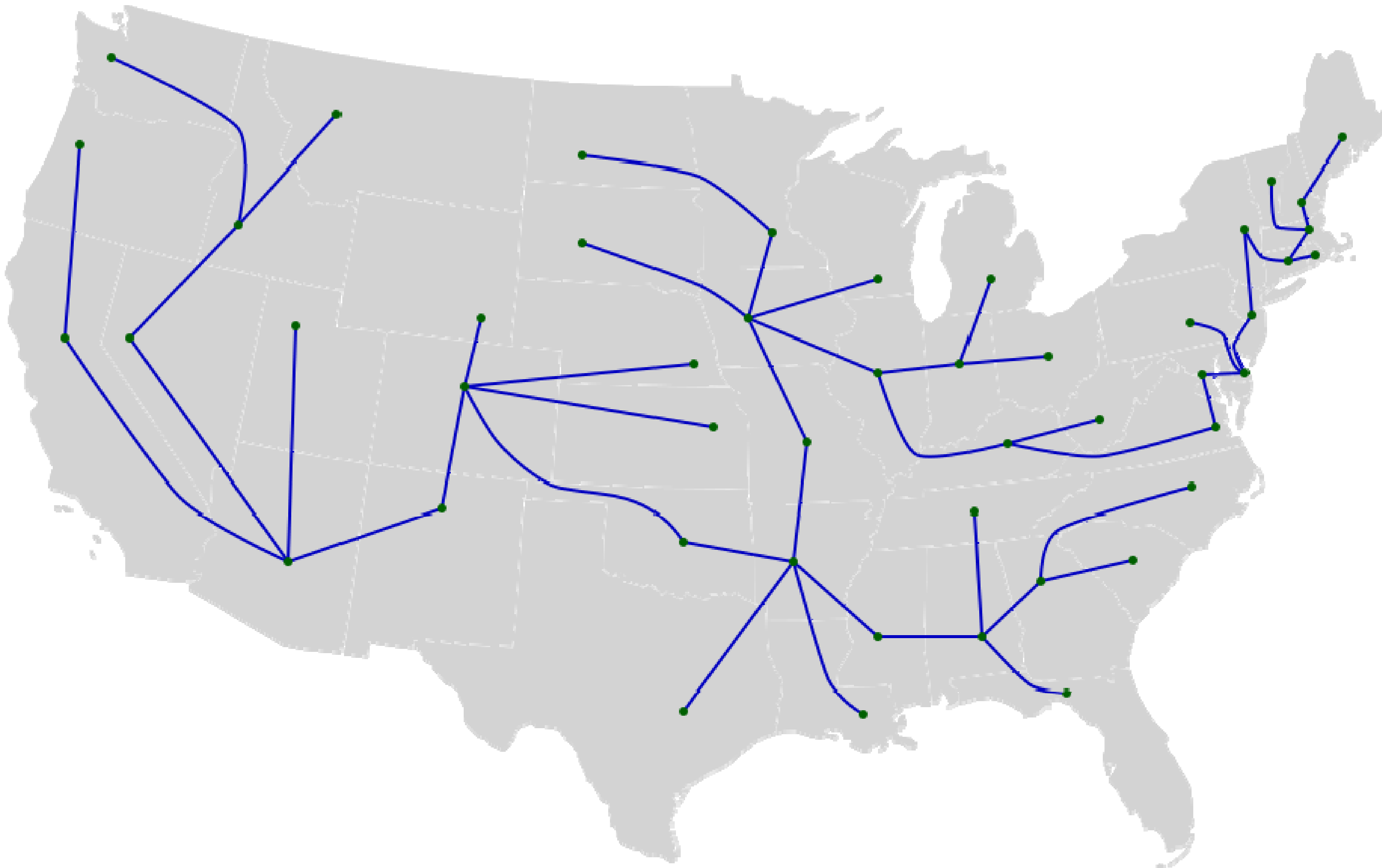
15,136 miles (24,359 km) total

68,656,026 possible tours

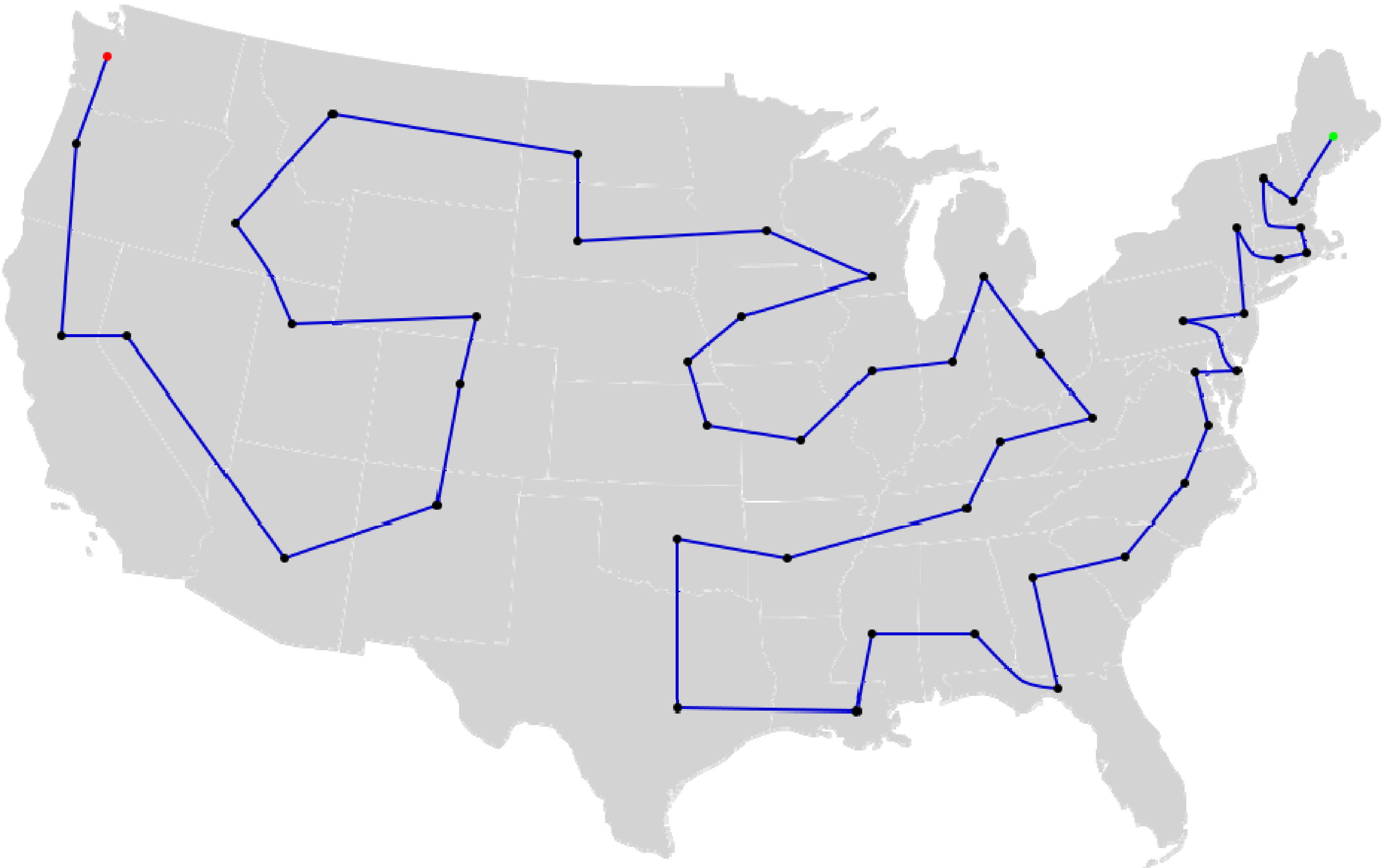
Limiting Node Degree



A Spanning Tree



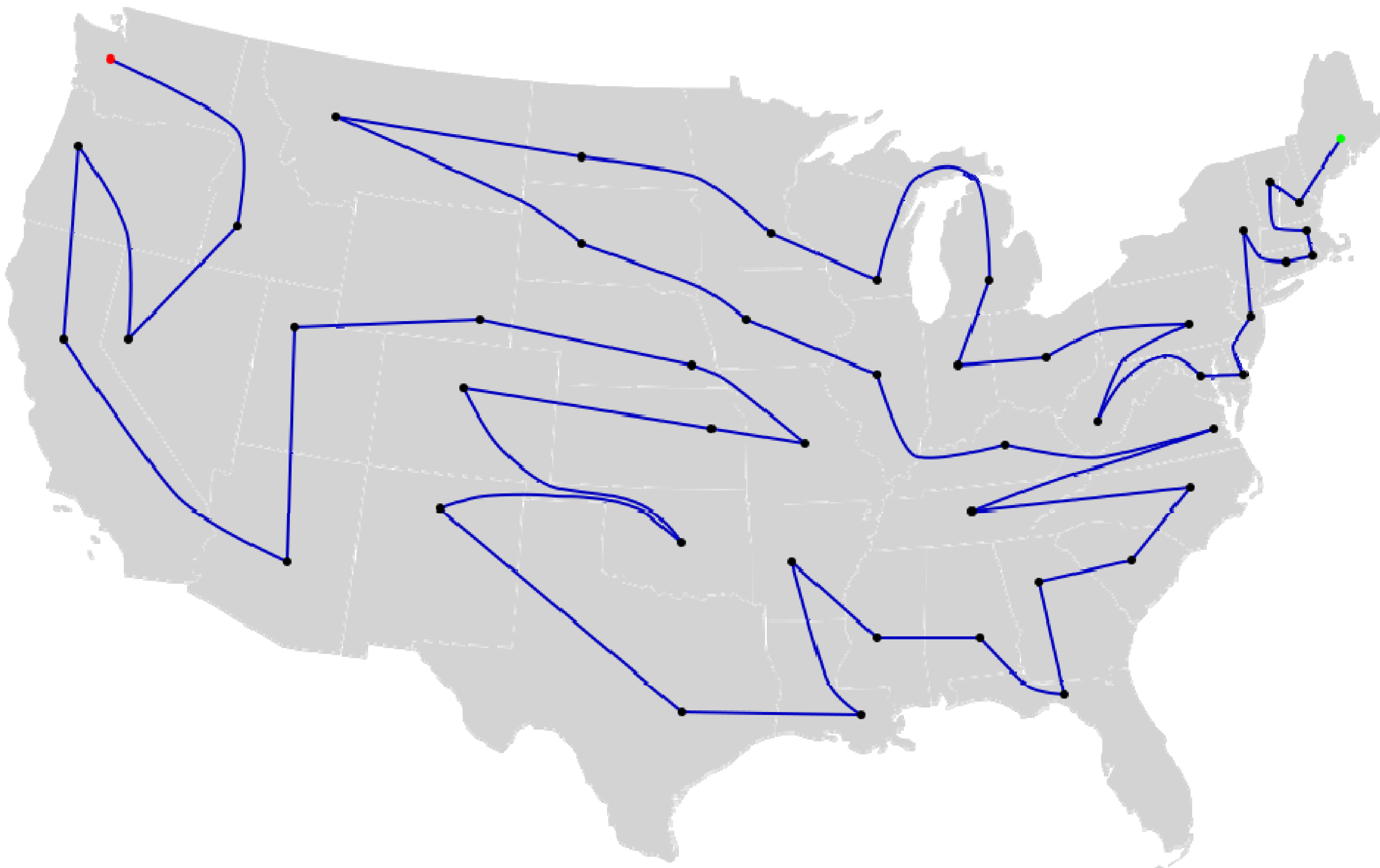
The Shortest Capitol Tour



A Traveling Salesman Path

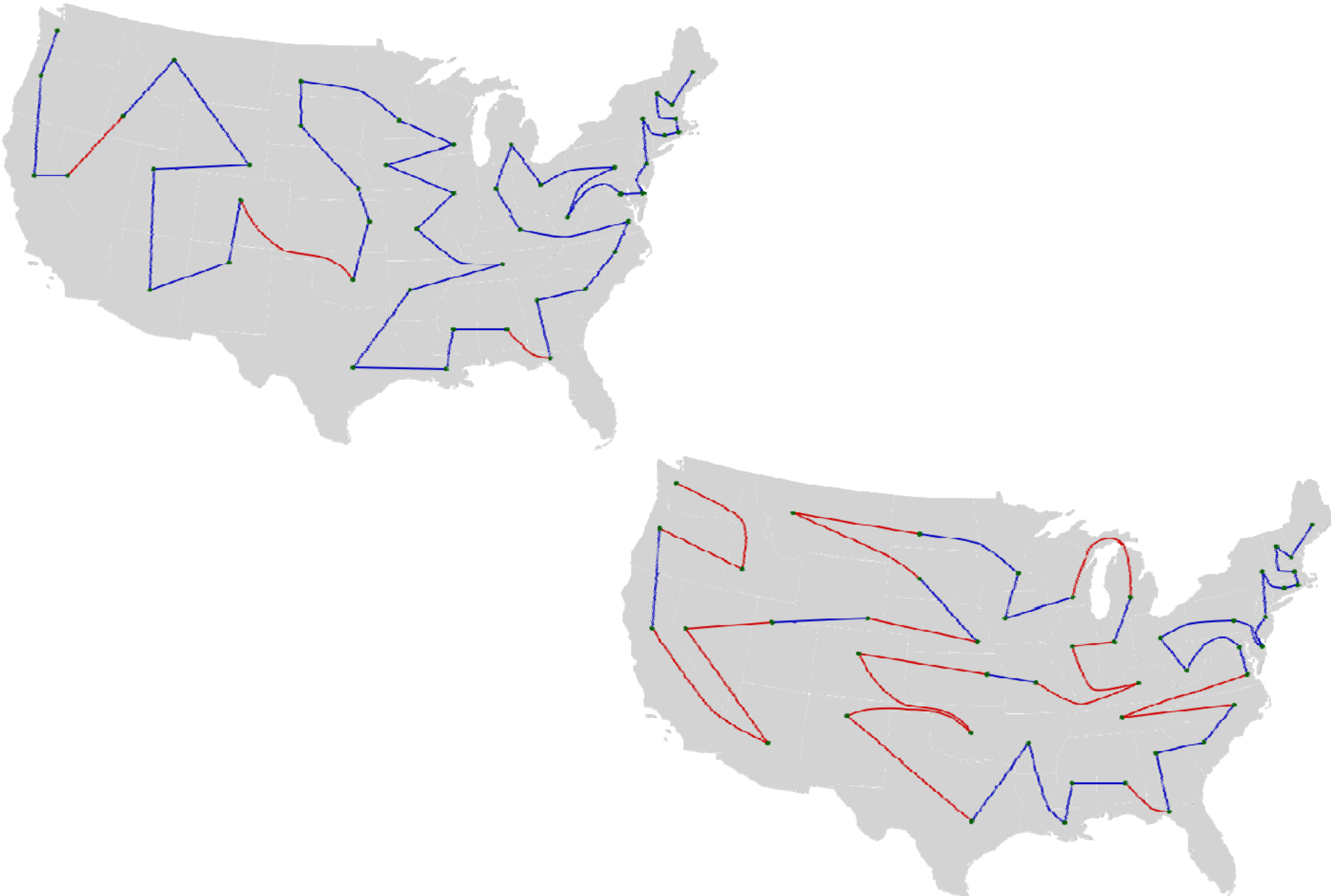
11,698 miles (18,826 km) total

The Longest Capitol Tour

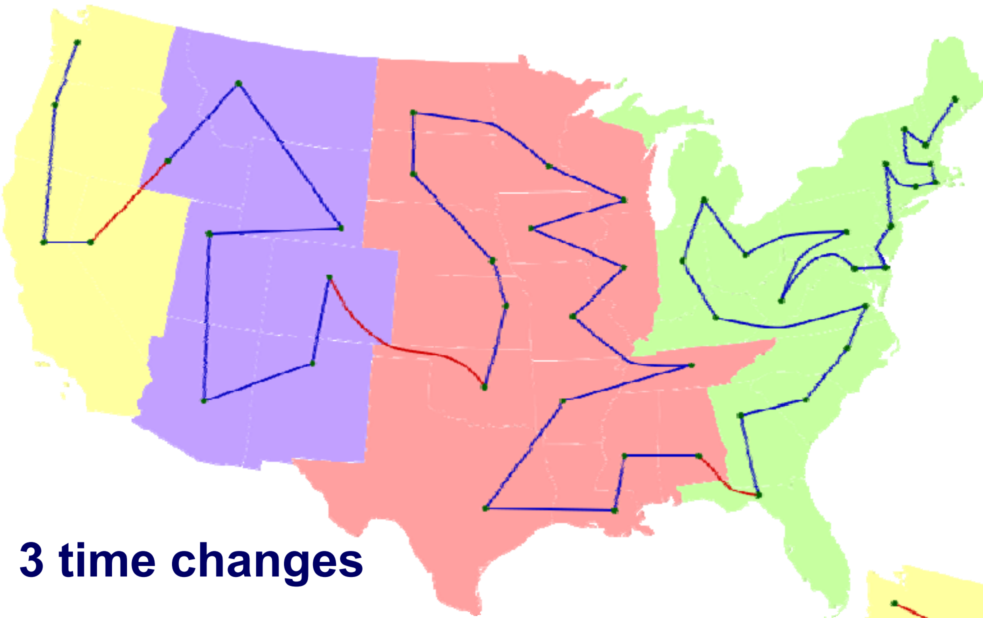


18,040 miles (29,033 km) total

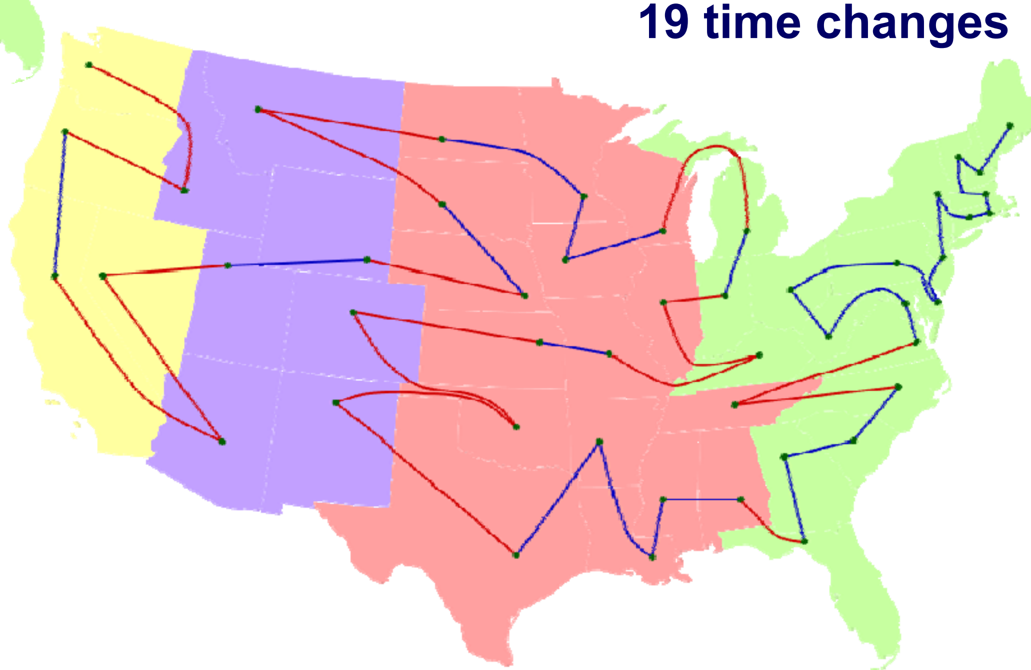
Two Interesting Capitol Tours



Two Interesting Capitol Tours

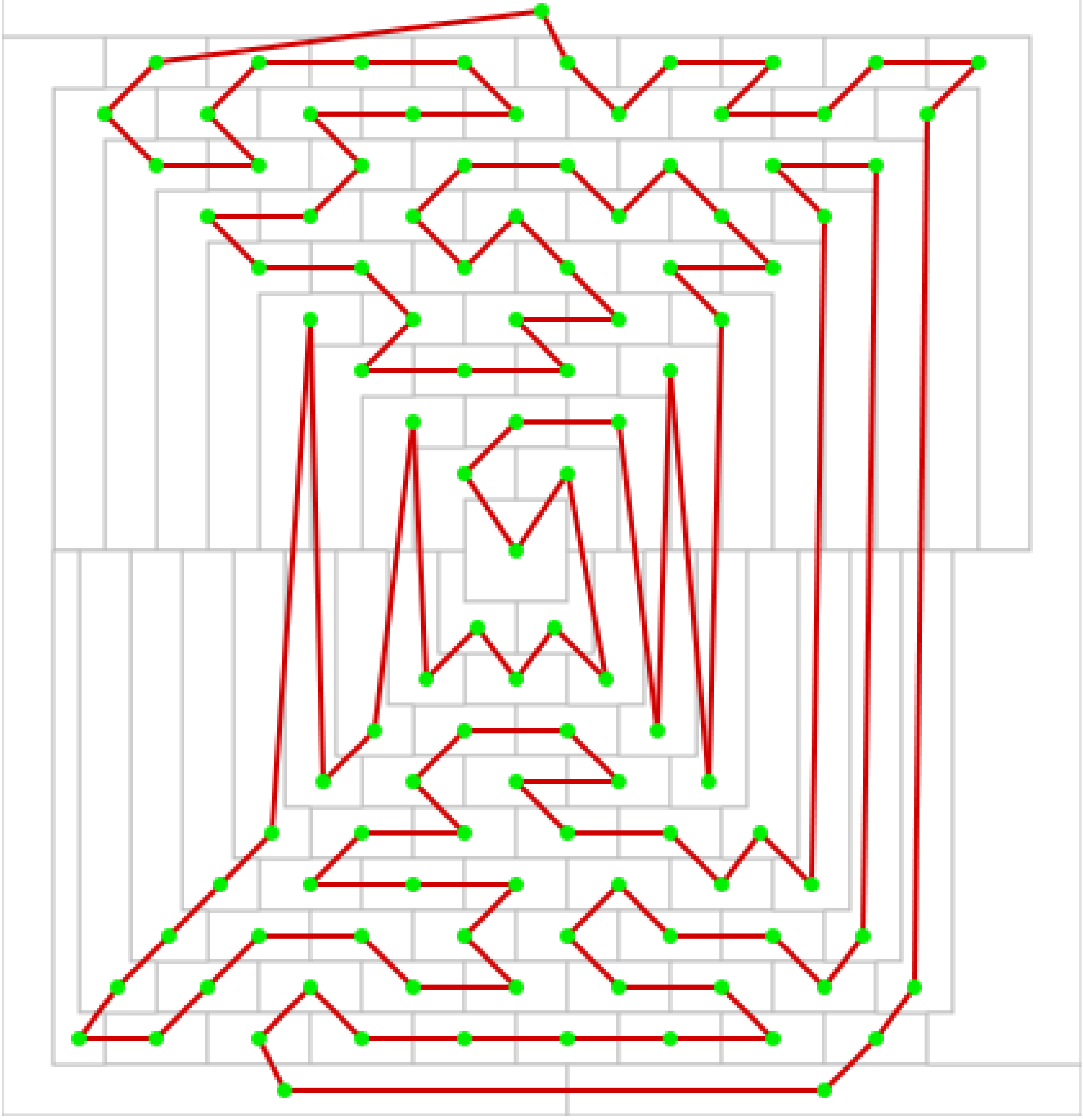


3 time changes



19 time changes

Touring MacGregor



A Hamiltonian Cycle

Lessons Learned

Graph Coloring

- Maps are a kind of graph
- Sudoku is a graph coloring problem

Hamiltonian Paths

- Find a path in graph that goes through every node once
- Considered a difficult problem

Boolean Methods

- Can encode wide variety of graph problems
- Can find solution using SAT solver
- In worst case, has exponential performance
  - But gets solution for many interesting problems