

15-462 Computer Graphics I  
Lecture 9

## Curves and Surfaces

Parametric Representations  
Cubic Polynomial Forms  
Hermite Curves  
Bezier Curves and Surfaces  
[Angel 10.1-10.6]

February 19, 2002  
Frank Pfenning  
Carnegie Mellon University

<http://www.cs.cmu.edu/~fp/courses/graphics/>

## Goals

- How do we draw surfaces?
  - Approximate with polygons
  - Draw polygons
- How do we specify a surface?
  - Explicit, implicit, parametric
- How do we approximate a surface?
  - Interpolation (use only points)
  - Hermite (use points and tangents)
  - Bezier (use points, and more points for tangents)
- Next lecture: splines, realization in OpenGL

## Explicit Representation

- Curve in 2D:  $y = f(x)$
- Curve in 3D:  $y = f(x), z = g(x)$
- Surface in 3D:  $z = f(x,y)$
- Problems:
  - How about a vertical line  $x = c$  as  $y = f(x)$ ?
  - Circle  $y = \pm (r^2 - x^2)^{1/2}$  two or zero values for  $x$
- Too dependent on coordinate system
- Rarely used in computer graphics

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## Implicit Representation

- Curve in 2D:  $f(x,y) = 0$ 
  - Line:  $ax + by + c = 0$
  - Circle:  $x^2 + y^2 - r^2 = 0$
- Surface in 3D:  $f(x,y,z) = 0$ 
  - Plane:  $ax + by + cz + d = 0$
  - Sphere:  $x^2 + y^2 + z^2 - r^2 = 0$
- $f(x,y,z)$  can describe 3D object:
  - Inside:  $f(x,y,z) < 0$
  - Surface:  $f(x,y,z) = 0$
  - Outside:  $f(x,y,z) > 0$

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## Algebraic Surfaces

- Special case of implicit representation
- $f(x,y,z)$  is polynomial in  $x, y, z$
- Quadrics: degree of polynomial  $\leq 2$
- Render more efficiently than arbitrary surfaces
- Implicit form often used in computer graphics
- How do we represent curves implicitly?

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## Parametric Form for Curves

- Curves: single parameter  $u$  (e.g. time)
- $x = x(u), y = y(u), z = z(u)$
- Circle:  $x = \cos(u), y = \sin(u), z = 0$
- Tangent described by derivative

$$\mathbf{p}(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} \quad \frac{d\mathbf{p}(u)}{du} = \begin{bmatrix} \frac{dx(u)}{du} \\ \frac{dy(u)}{du} \\ \frac{dz(u)}{du} \end{bmatrix}$$

- Magnitude is “velocity”

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## Parametric Form for Surfaces

- Use parameters  $u$  and  $v$
- $x = x(u,v)$ ,  $y = y(u,v)$ ,  $z = z(u,v)$
- Describes surface as both  $u$  and  $v$  vary
- Partial derivatives describe tangent plane at each point  $\mathbf{p}(u,v) = [x(u,v) \ y(u,v) \ z(u,v)]^T$

$$\frac{\partial \mathbf{p}(u,v)}{\partial u} = \begin{bmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{bmatrix} \quad \frac{\partial \mathbf{p}(u,v)}{\partial v} = \begin{bmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{bmatrix}$$

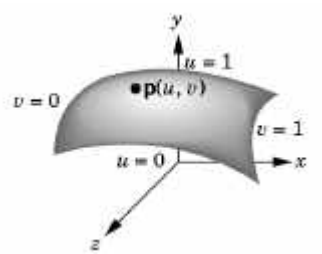
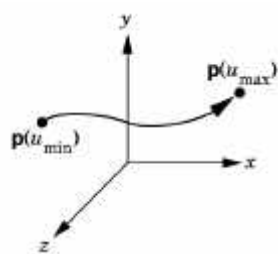
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## Assessment of Parametric Forms

- Parameters often have natural meaning
- Easy to define and calculate
  - Tangent and normal
  - Curves segments (for example,  $0 \leq u \leq 1$ )
  - Surface patches (for example,  $0 \leq u,v \leq 1$ )



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## Parametric Polynomial Curves

- Restrict  $x(u)$ ,  $y(u)$ ,  $z(u)$  to be polynomial in  $u$

- Fix degree  $n$
- $$p(u) = \sum_{k=0}^n c_k u^k$$

- Each  $c_k$  is a column vector

$$c_k = \begin{bmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{bmatrix}$$

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## Parametric Polynomial Surfaces

- Restrict  $x(u,v)$ ,  $y(u,v)$ ,  $z(u,v)$  to be polynomial of fixed degree  $n$

$$p(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix} = \sum_{i=0}^n \sum_{k=0}^n c_{ik} u^i v^k$$

- Each  $c_{ik}$  is a 3-element column vector
- Restrict to simple case where  $0 \leq u, v \leq 1$

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## Approximating Surfaces

- Use parametric polynomial surfaces
- Important concepts:
  - Join points for segments and patches
  - Control points to interpolate
  - Tangents and smoothness
  - Blending functions to describe interpolation
- First curves, then surfaces



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## Outline

- Parametric Representations
- Cubic Polynomial Forms
- Hermite Curves
- Bezier Curves and Surfaces

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## Cubic Polynomial Form

- Degree 3 appears to be a useful compromise
- Curves:

$$p(u) = c_0 + c_1u + c_2u^2 + c_3u^3 = \sum_{k=0}^3 c_k u^k$$

- Each  $c_k$  is a column vector  $[c_{kx} \ c_{ky} \ c_{kz}]^T$
- From control information (points, tangents) derive 12 values  $c_{kx}, c_{ky}, c_{kz}$  for  $0 \leq k \leq 3$
- These determine cubic polynomial form
- Later: how to render

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## Interpolation by Cubic Polynomials

- Simplest case, although rarely used
- Curves:
  - Given 4 control points  $p_0, p_1, p_2, p_3$
  - All should lie on curve: 12 conditions, 12 unknowns
- Space  $0 \leq u \leq 1$  evenly  
 $p_0 = p(0), p_1 = p(1/3), p_2 = p(2/3), p_3 = p(1)$

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## Equations to Determine $c_k$

- Plug in values for  $u = 0, 1/3, 2/3, 1$

$$p_0 = p(0) = c_0$$

$$p_1 = p\left(\frac{1}{3}\right) = c_0 + \frac{1}{3}c_1 + \left(\frac{1}{3}\right)^2c_2 + \left(\frac{1}{3}\right)^3c_3$$

$$p_2 = p\left(\frac{2}{3}\right) = c_0 + \frac{2}{3}c_1 + \left(\frac{2}{3}\right)^2c_2 + \left(\frac{2}{3}\right)^3c_3$$

$$p_3 = p(1) = c_0 + c_1 + c_2 + c_3$$

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \frac{2}{3} & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Note:  
 $p_k$  and  $c_k$   
are vectors!

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## Interpolating Geometry Matrix

- Invert  $A$  to obtain interpolating geometry matrix

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & 4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix} \quad c = A^{-1}p$$

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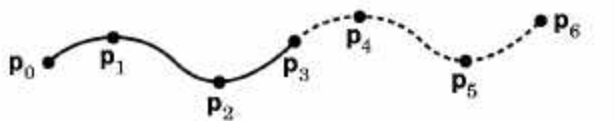
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## Joining Interpolating Segments

- Do not solve degree  $n$  for  $n$  points
- Divide into overlap sequences of 4 points
- $p_0, p_1, p_2, p_3$  then  $p_3, p_4, p_5, p_6$ , etc.



- At join points
  - Will be continuous ( $C^0$  continuity)
  - Derivatives will usually not match (no  $C^1$  continuity)

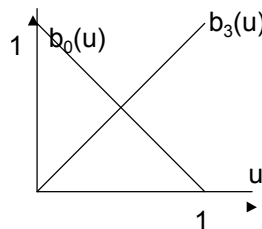
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## Blending Functions

- Make explicit, how control points contribute
- Simplest example: straight line with control points  $p_0$  and  $p_3$
- $p(u) = (1 - u) p_0 + u p_3$
- $b_0(u) = 1 - u$ ,  $b_3(u) = u$



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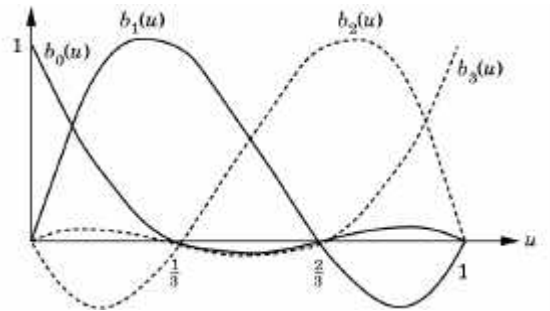
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## Blending Polynomials for Interpolation

- Each blending polynomial is a cubic
- Solve (see [Angel, p. 427]):

$$p(u) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_2 + b_3(u)p_3$$



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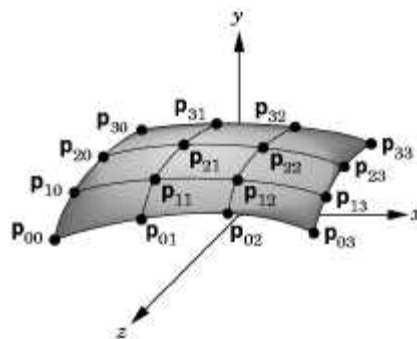
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## Cubic Interpolation Patch

- Bicubic surface patch with  $4 \times 4$  control points

$$p(u) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_2 + b_3(u)p_3$$

Note: each  $c_{ik}$  is  
3 column vector  
(48 unknowns)



[Angel, Ch. 10.4.2]

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## Outline

- Parametric Representations
- Cubic Polynomial Forms
- Hermite Curves
- Bezier Curves and Surfaces

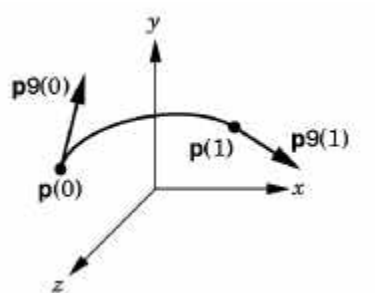
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## Hermite Curves

- Another cubic polynomial curve
- Specify two endpoints and their tangents



[diagram correction  $p9 = p'$ ]

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## Deriving the Hermite Form

- As before

$$p(0) = p_0 = c_0$$

$$p(1) = p_3 = c_0 + c_1 + c_2 + c_3$$

- Calculate derivative

$$p'(u) = \begin{bmatrix} \frac{dx}{du} \\ \frac{dy}{du} \\ \frac{dz}{du} \end{bmatrix} = c_1 + 2uc_2 + 3u^2c_3$$

- Yields

$$p'_0 = p'(0) = c_1$$

$$p'_3 = p'(1) = c_1 + 2c_2 + 3c_3$$

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## Summary of Hermite Equations

- Write in matrix form
- Remember  $p_k$  and  $p'_k$  and  $c_k$  are vectors!

$$\begin{bmatrix} p_0 \\ p_3 \\ p'_0 \\ p'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Let  $q = [p_0 \ p_3 \ p'_0 \ p'_3]^T$  and invert to find Hermite geometry matrix  $M_H$  satisfying

$$c = M_H q$$

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## Blending Functions

- Explicit Hermite geometry matrix

$$M_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

- Blending functions for  $u = [1 \ u \ u^2 \ u^3]^T$

$$b(u) = M_H^T u = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}$$

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## Join Points for Hermite Curves

- Match points and tangents (derivates)

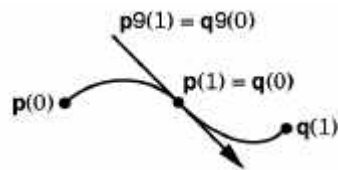


Diagram correction  
[ $p' = p'$ ,  $q' = q'$ ]

- Much smoother than point interpolation
- How to obtain the tangents?
- Skip Hermite surface patch
- More widely used: Bezier curves and surfaces

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## Parametric Continuity

- Matching endpoints ( $C^0$  parametric continuity)

$$\mathbf{p}(1) = \begin{bmatrix} p_x(1) \\ p_y(1) \\ p_z(1) \end{bmatrix} = \begin{bmatrix} q_x(0) \\ q_y(0) \\ q_z(0) \end{bmatrix} = \mathbf{q}(0)$$

- Matching derivatives ( $C^1$  parametric continuity)

$$\mathbf{p}'(1) = \begin{bmatrix} p'_x(1) \\ p'_y(1) \\ p'_z(1) \end{bmatrix} = \begin{bmatrix} q'_x(0) \\ q'_y(0) \\ q'_z(0) \end{bmatrix} = \mathbf{q}'(0)$$

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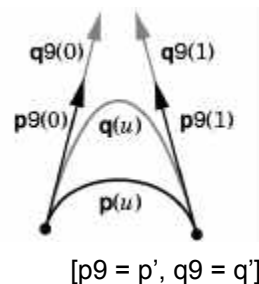
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## Geometric Continuity

- For matching tangents, less is required

$$\mathbf{p}'(1) = \begin{bmatrix} p'_x(1) \\ p'_y(1) \\ p'_z(1) \end{bmatrix} = k \begin{bmatrix} q'_x(0) \\ q'_y(0) \\ q'_z(0) \end{bmatrix} = k\mathbf{q}'(0)$$

- $G^1$  geometric continuity
- Extends to higher derivatives



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## Outline

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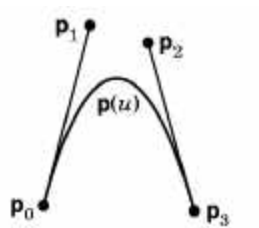
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## Bezier Curves

- Widely used in computer graphics
- Approximate tangents by using control points

$$p'(0) = 3(p_1 - p_0)$$

$$p'(1) = 3(p_3 - p_2)$$



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## Equations for Bezier Curves

- Set up equations for cubic parametric curve
- Recall:

$$\begin{aligned}p(u) &= c_0 + c_1u + c_2u^2 + c_3u^3 \\p'(u) &= c_1 + 2c_2u + 3c_3u^2\end{aligned}$$

- Solve for  $c_k$

$$\begin{aligned}p_0 &= p(0) = c_0 \\p_3 &= p(1) = c_0 + c_1 + c_2 + c_3 \\p'(0) &= 3p_1 - 3p_0 = c_1 \\p'(1) &= 3p_3 - 3p_2 = c_1 + 2c_2 + 3c_3\end{aligned}$$

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## Bezier Geometry Matrix

- Calculate Bezier geometry matrix  $M_B$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = M_B \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \text{ so } M_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

- Have  $C^0$  continuity, not  $C^1$  continuity
- Have  $C^1$  continuity with additional condition

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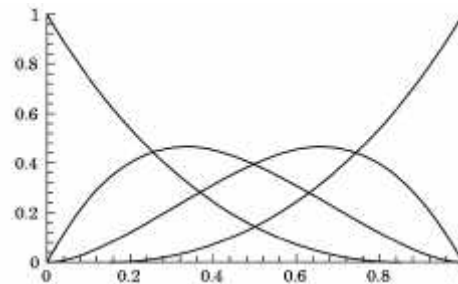


## Blending Polynomials

- Determine contribution of each control point

$$\mathbf{b}(u) = \mathbf{M}_B^T \mathbf{u} = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}$$

Smooth blending  
polynomials



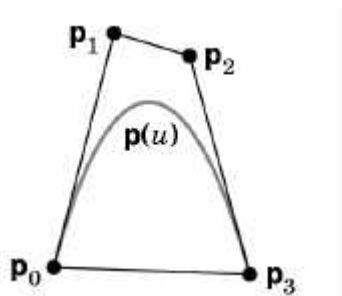
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## Convex Hull Property

- Bezier curve contained entirely in convex hull of control points
- Determined choice of tangent coefficient (?)



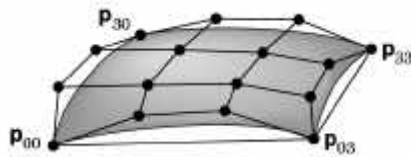
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## Bezier Surface Patches

- Specify Bezier patch with  $4 \times 4$  control points



- Bezier curves along the boundary

$$\mathbf{p}(0, 0) = \mathbf{p}_{00}$$

$$\frac{\partial \mathbf{p}}{\partial u}(0, 0) = 3(\mathbf{p}_{10} - \mathbf{p}_{00})$$

$$\frac{\partial \mathbf{p}}{\partial v}(0, 0) = 3(\mathbf{p}_{01} - \mathbf{p}_{00})$$

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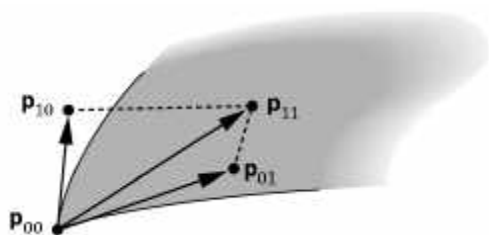
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## Twist

- Inner points determine twist at corner

$$\frac{\partial^2 \mathbf{p}}{\partial u \partial v}(0, 0) = 9(\mathbf{p}_{00} - \mathbf{p}_{01} + \mathbf{p}_{10} - \mathbf{p}_{11})$$

- Flat means  $\mathbf{p}_{00}, \mathbf{p}_{10}, \mathbf{p}_{01}, \mathbf{p}_{11}$  in one plane
- $(\partial^2 \mathbf{p} / \partial u \partial v)(0, 0) = 0$



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## Summary

- Parametric Representations
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## Preview

- B-Splines: more continuity ( $C^2$ )
- Non-uniform B-splines (“heavier” points)
- Non-uniform rational B-splines (NURBS)
  - Rational functions instead of polynomials
  - Based on homogeneous coordinates
- Rendering and recursive subdivision
- Curves and surfaces in OpenGL

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## Announcements

- Still have some graded homeworks (Asst. 2)
- Model solution coming soon
- Assignment 3 due Thursday before midnight
- Assignment 4 on curves and surfaces out Thursday