Chapter 3

Sequent Calculus

In this chapter we develop the sequent calculus as a formal system for proof search in natural deduction. The sequent calculus was originally introduced by Gentzen [Gen35], primarily as a technical device for proving consistency of predicate logic. Our goal of describing a proof search procedure for natural deduction predisposes us to a formulation due to Kleene [Kle52] called G_3.

We introduce the sequent calculus in two steps. The first step is based on the simple strategy of building a natural deduction by using introduction rules bottom-up and elimination rules top-down. The result is an intercalation calculus which applies both to intuitionistic and classical logic [Byr99]. The second step consists of reformulating the rules for intercalation so that both forms of rules work bottom-up, resulting in the sequent calculus.

We also show how intercalation derivations lead to more compact proof terms, and how to extract proof terms from sequent calculus derivations.

3.1 Intercalation

A simple strategy in the search for a natural deduction is to use introduction rules reasoning bottom-up (from the proposed theorem towards the hypotheses) and the elimination rules top-down (from the assumptions towards the proposed theorem). When they meet in the middle we have found a *normal* deduction. Towards the end of this chapter we show that this strategy is in fact complete: if a proposition A has a natural deduction then it has a normal deduction. First, however, we need to make this strategy precise.

A general technique for representing proof search strategies is to introduce new judgments which permit only those derivations which can be found by the intended strategy. We then prove the correctness of the new, restricted judgments by appropriate soundness and completeness theorems.

In this case, we introduce two judgments:

$A \Uparrow$ Proposition A has a normal deduction, and

$A \downarrow$ Proposition A is extracted from a hypothesis.

They are defined by restricting the rules of natural deduction according to their status as introduction or elimination rules. Hypotheses can be trivially extracted. Therefore the necessary hypothetical judgments (in localized form, see Section 2.3) are

$u_1{:}A_1 \downarrow, \ldots, u_n{:}A_n \downarrow \;\vdash A \Uparrow$ and

$u_1{:}A_1 \downarrow, \ldots, u_n{:}A_n \downarrow \;\vdash A \downarrow$.

We write $\Gamma^\downarrow$ for a context of the form shown above.

Hypotheses. The general rule for hypotheses simply reflects the nature of hypothetical judgments.

$$\frac{}{\Gamma_1^\downarrow, u{:}A \downarrow, \Gamma_2^\downarrow \vdash A \downarrow}\, u$$

Coercion. The bottom-up and top-down derivations must be able to meet in the middle.

$$\frac{\Gamma^\downarrow \vdash A \downarrow}{\Gamma^\downarrow \vdash A \Uparrow}\, {\downarrow}{\Uparrow}$$

Looked at another way, this rule allows us to coerce any extraction derivation to a normal deduction. Of course, the opposite coercion would contradict the intended strategy.

Conjunction. The rules for conjunction exhibit no unexpected features: the introduction rule is classified as a bottom-up rule, the elimination rule is classified as a top-down rule.

$$\frac{\Gamma^\downarrow \vdash A \Uparrow \qquad \Gamma^\downarrow \vdash B \Uparrow}{\Gamma^\downarrow \vdash A \wedge B \Uparrow}\, {\wedge}\mathrm{I}$$

$$\frac{\Gamma^\downarrow \vdash A \wedge B \downarrow}{\Gamma^\downarrow \vdash A \downarrow}\, {\wedge}\mathrm{E_L} \qquad \frac{\Gamma^\downarrow \vdash A \wedge B \downarrow}{\Gamma^\downarrow \vdash B \downarrow}\, {\wedge}\mathrm{E_R}$$

Truth. For truth, there is only an introduction rule which is classified as normal.

$$\frac{}{\Gamma^\downarrow \vdash \top \Uparrow}\, {\top}\mathrm{I}$$

Implication. The introduction rule for implication is straightforward. In the elimination rule we require that the the second premise is normal. It is only the first premise (whose primary connective is eliminated in this rule) which must be extracted from a hypothesis.

$$\frac{\Gamma^\downarrow, u{:}A \downarrow \;\vdash B \Uparrow}{\Gamma^\downarrow \vdash A \supset B \Uparrow}\, {\supset}\mathrm{I}^u \qquad \frac{\Gamma^\downarrow \vdash A \supset B \downarrow \qquad \Gamma^\downarrow \vdash A \Uparrow}{\Gamma^\downarrow \vdash B \downarrow}\, {\supset}\mathrm{E}$$

Disjunction. The introduction rules for disjunction are straightforward. For the elimination rule, again the premise with the connective which is eliminated must have a top-down derivation. The new assumptions in each branch also are top-down derivations. Overall, for the derivation to be normal we must require the derivations of both premises to be normal.

$$\frac{\Gamma^{\downarrow} \vdash A \Uparrow}{\Gamma^{\downarrow} \vdash A \vee B \Uparrow} \vee\mathrm{I_L} \qquad \frac{\Gamma^{\downarrow} \vdash B \Uparrow}{\Gamma^{\downarrow} \vdash A \vee B \Uparrow} \vee\mathrm{I_R}$$

$$\frac{\Gamma^{\downarrow} \vdash A \vee B \downarrow \qquad \Gamma^{\downarrow}, u{:}A \downarrow\ \vdash C \Uparrow \qquad \Gamma^{\downarrow}, w{:}B \downarrow\ \vdash C \Uparrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \vee\mathrm{E}^{u,w}$$

It would also be consistent to allow the derivations of C to be extractions, but it is not necessary to obtain a complete search procedure and complicates the relation to the sequent calculus (see Exercise 3.1).

Falsehood. Falsehood corresponds to a disjunction with no alternatives. Therefore there is no introduction rule, and the elimination rule has no cases. This consideration yields

$$\frac{\Gamma^{\downarrow} \vdash \bot \downarrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \bot\mathrm{E}.$$

For this rule, it does not appear to make sense to allow the conclusion as having been constructed top-down, since the proposition C would be completely unrestricted.

Negation. Negation combines elements from implication and falsehood, since we may think of $\neg A$ as $A \supset \bot$.

$$\frac{\Gamma^{\downarrow}, u{:}A \downarrow\ \vdash p \Uparrow}{\Gamma^{\downarrow} \vdash \neg A \Uparrow} \neg\mathrm{I}^{p,u} \qquad \frac{\Gamma^{\downarrow} \vdash \neg A \downarrow \qquad \Gamma^{\downarrow} \vdash A \Uparrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \neg\mathrm{E}$$

Universal Quantification. Universal quantification does not introduce any new considerations.

$$\frac{\Gamma^{\downarrow} \vdash [a/x]A \Uparrow}{\Gamma^{\downarrow} \vdash \forall x.\ A \Uparrow} \forall\mathrm{I}^{a} \qquad \frac{\Gamma^{\downarrow} \vdash \forall x.\ A \downarrow}{\Gamma^{\downarrow} \vdash [t/x]A \downarrow} \forall\mathrm{E}$$

Existential Quantification. Existential quantification is similar to disjunction and a more lenient view of extraction is possible here, too (see Exercise 3.1).

$$\frac{\Gamma^{\downarrow} \vdash [t/x]A \Uparrow}{\Gamma^{\downarrow} \vdash \exists x.\ A \Uparrow} \exists\mathrm{I} \qquad \frac{\Gamma^{\downarrow} \vdash \exists x.\ A \downarrow \qquad \Gamma^{\downarrow}, u{:}[a/x]A \downarrow\ \vdash C \Uparrow}{\Gamma^{\downarrow} \vdash C \Uparrow} \exists\mathrm{E}^{a,u}$$

It is quite easy to see that normal and extraction derivations are sound with respect to natural deduction. In order to state and prove this theorem, we introduce some conventions. Given a context

$$\Gamma^{\downarrow} = u_1{:}A_1 \downarrow, \ldots, u_n{:}A_n \downarrow$$

we denote

$$u_1{:}A_1, \ldots, u_n{:}A_n$$

by Γ and vice versa.

Theorem 3.1 (Soundness of Normal Deductions)

1. *If $\Gamma^{\downarrow} \vdash A \Uparrow$ then $\Gamma \vdash A$, and*
2. *if $\Gamma^{\downarrow} \vdash A \downarrow$ then $\Gamma \vdash A$.*

Proof: By induction on the structure of the given derivations. We show only three cases, since the proof is absolutely straightforward.

Case:

$$\mathcal{E} = \dfrac{}{\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash A \downarrow}\, u$$

The we construct directly $\Gamma_1, u{:}A, \Gamma_2 \vdash A$.

Case:

$$\mathcal{N} = \dfrac{\begin{array}{c}\mathcal{E}\\ \Gamma^{\downarrow} \vdash A \downarrow\end{array}}{\Gamma^{\downarrow} \vdash A \Uparrow}\, {\downarrow}{\Uparrow}$$

Then $\Gamma \vdash A$ by induction hypothesis on $\mathcal{E}$.

Case:

$$\mathcal{N} = \dfrac{\begin{array}{c}\mathcal{N}_2\\ \Gamma^{\downarrow}, u{:}A_1 \downarrow\ \vdash A_2 \Uparrow\end{array}}{\Gamma^{\downarrow} \vdash A_1 \supset A_2 \Uparrow}\, {\supset}\mathrm{I}^u$$

$\Gamma, u{:}A_1 \vdash A_2$	By i.h. on $\mathcal{N}_2$
$\Gamma \vdash A_1 \supset A_2$	By rule $\supset$I

□

When trying to give a translation in the other direction we encounter a difficulty: certain patterns of inference cannot be annotated directly. For example, consider

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}\, \wedge\mathrm{I}}{\Gamma \vdash A}\, \wedge\mathrm{E_L}.$$

If we try to classify each judgment, we obtain a conflict:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma \vdash A \Uparrow\end{array} \qquad \begin{array}{c}\mathcal{E}'\\ \Gamma \vdash B \Uparrow\end{array}}{\Gamma \vdash A \wedge B\ ?}\wedge\mathrm{I}}{\Gamma \vdash A \downarrow}\wedge\mathrm{E_L}.$$

In this particular case, we can avoid the conflict: in order to obtain the derivation of $A \Uparrow$ we can just translate the derivation $\mathcal{D}$ and avoid the final two inferences! In general, we can try to apply local reductions to the given original derivation until no situations of the form above remain. This approach is called *normalization*. It is not easy to prove that normalization terminates, and the situation is complicated by the fact that the local reductions alone do not suffice to transform an arbitrary natural deduction into normal form (see Exercise 3.2).

Here, we follow an alternative approach to prove completeness of normal deductions. First, we temporarily augment the system with another rule which makes the translation from natural deductions immediate. Then we relate the resulting system to a sequent calculus and show that the additional rule was redundant.

A candidate for the additional rule is easy to spot: we just add the missing coercion from normal to extraction deductions. Since all rules are present, we can just coerce back and forth as necessary in order to obtain a counterpart for any natural deduction in this extended system. Of course, the resulting derivations are no longer normal, which we indicate by decorating the turnstile with a "+". The judgments $\Gamma^{\downarrow} \vdash^{+} A \Uparrow$ and $\Gamma^{\downarrow} \vdash^{+} A \downarrow$ are defined by all counterparts of all rules which define normal and extracting derivations, plus the rule

$$\dfrac{\Gamma^{\downarrow} \vdash^{+} A \Uparrow}{\Gamma^{\downarrow} \vdash^{+} A \downarrow}\Uparrow\downarrow$$

Now the annotation in the example above can be completed.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma \vdash^{+} A \Uparrow\end{array} \qquad \begin{array}{c}\mathcal{E}'\\ \Gamma \vdash^{+} B \Uparrow\end{array}}{\Gamma \vdash^{+} A \wedge B \Uparrow}\wedge\mathrm{I}}{\Gamma \vdash^{+} A \wedge B \downarrow}\Uparrow\downarrow}{\Gamma \vdash^{+} A \downarrow}\wedge\mathrm{E_L}$$

Both soundness and completeness of the extended calculus with respect to natural deduction is easy to see.

Theorem 3.2 (Soundness of Annotated Deductions)

1. *If* $\Gamma^{\downarrow} \vdash^{+} A \Uparrow$ *then* $\Gamma \vdash A$*, and*

2. *if* $\Gamma^{\downarrow} \vdash^{+} A \downarrow$ *then* $\Gamma \vdash A$.

Proof: By simultaneous induction over the structure of the given derivations. □

The constructive proof of the completeness theorem below will contain an algorithm for annotating a given natural deduction.

Theorem 3.3 (Completeness of Annotated Deductions)

1. *If* $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash^{+} A \Uparrow$, *and*
2. *if* $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash^{+} A \downarrow$.

Proof: By induction over the structure of the given derivation. We show only two cases.

Case:

$$\mathcal{D} = \cfrac{\begin{matrix}\mathcal{D} \\ \Gamma \vdash B \supset A\end{matrix} \qquad \begin{matrix}\mathcal{E} \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A} \supset\!\mathrm{E}$$

$\Gamma^{\downarrow} \vdash^{+} B \supset A \downarrow$	By i.h. (2) on $\mathcal{D}$
$\Gamma^{\downarrow} \vdash^{+} B \Uparrow$	By i.h. (1) on $\mathcal{E}$
$\Gamma^{\downarrow} \vdash^{+} A \downarrow$	By rule $\supset$E, proving (2)
$\Gamma^{\downarrow} \vdash^{+} A \Uparrow$	By rule $\downarrow\Uparrow$, proving (1)

Case:

$$\mathcal{D} = \cfrac{\begin{matrix}\mathcal{D}_2 \\ \Gamma, u{:}A_1 \vdash A_2\end{matrix}}{\Gamma \vdash A_1 \supset A_2} \supset\!\mathrm{I}^u$$

$\Gamma^{\downarrow}, u{:}A_1 \downarrow \vdash^{+} A_2 \Uparrow$	By i.h. (1) on $\mathcal{D}_2$
$\Gamma^{\downarrow} \vdash^{+} A_1 \supset A_2 \Uparrow$	By rule $\supset$I^u, proving (1)
$\Gamma^{\downarrow} \vdash^{+} A_1 \supset A_2 \downarrow$	By rule $\Uparrow\downarrow$, proving (2)

□

Even though natural deductions and annotated deductions are very similar, they are not in bijective correspondence. For example, in an annotated deduction we can simply alternate the two coercions an arbitrary number of times. Under the translation to natural deduction, all of these are identified.

Before we introduce the sequent calculus, we make a brief excursion to study the impact of annotations on proof terms.

3.2 Compact Proof Terms

The proof terms introduced in Section 2.4 sometimes contain significant amounts of redundant information. The reason are the propositions which label λ-abstractions and also occur in the inl^A, inr^A, $\mu^p u{:}A$, $\cdot_A$, and abort^A constructs. For example, assume we are given a proof term $\lambda u{:}A.\ M$ and we are supposed to check if it represents a proof of $A' \supset B$. We then have to check that $A = A'$ and, moreover, the information is duplicated. The reason for this duplication was the intended invariant that every term proves a unique proposition. Under the interpretations of propositions as types, this means we can always synthesize a unique type for every valid term. However, we can improve this if we alternate between synthesizing a type and checking a term against a given type.

Therefore we introduce two classes of terms: those whose type can be synthesized, and those which can be checked against a type. Interestingly, this corresponds precisely with the annotations as introduction or elimination rules given above. We ignore negation again, thinking of $\neg A$ as $A \supset \bot$. We already discussed why the eliminations for disjunction and falsehood appear among the intro terms.

Intro Terms	I	$::=$	$\langle I_1, I_2 \rangle$	Conjunction
			$\mid \lambda u.\ I$	Implication
			$\mid \mathrm{inl}\, I \mid \mathrm{inr}\, I$	Disjunction
			$\mid (\,\mathbf{case}\ E\ \mathbf{of}\ \mathrm{inl}\, u_1 \Rightarrow I_1 \mid \mathrm{inr}\, u_2 \Rightarrow I_2)$	
			$\mid \langle\,\rangle$	Truth
			$\mid \mathrm{abort}\, E$	Falsehood
			$\mid E$	Coercion
Elim Terms	E	$::=$	u	Hypotheses
			$\mid E\, I$	Implication
			$\mid \mathrm{fst}\, E \mid \mathrm{snd}\, E$	Conjunction
			$\mid (I : A)$	Coercion

The presence of E as an intro term corresponds to the coercion $\downarrow\Uparrow$ which is present in normal deductions. The presence of $(I : A)$ as an elim term corresponds to the coercion $\Uparrow\downarrow$ which is present only in the extended system. Therefore, a normal deduction can be represented without any internal type information, while a general deduction requires information at the point where an introduction rule is directly followed by an elimination rule. It is easy to endow the annotated natural deduction judgments with the modified proof terms from above. We leave the details to Exercise 3.3. The two judgments are $\Gamma^\downarrow \vdash^+ I : A \Uparrow$ and $\Gamma^\downarrow \vdash^+ E : A \downarrow$.

Now we can prove the correctness of bi-directional type-checking.

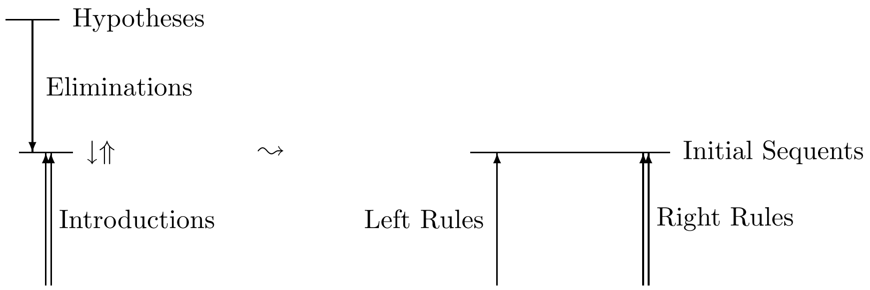
Theorem 3.4 (Bi-Directional Type-Checking)

1. *Given $\Gamma^\downarrow$, I, and A. Then either $\Gamma^\downarrow \vdash^+ I : A \Uparrow$ or not.*
2. *Given $\Gamma^\downarrow$ and E. Then either there is a unique A such that $\Gamma^\downarrow \vdash^+ E : A \downarrow$ or there is no such A.*

Proof: See Exercise 3.3. □

3.3 Sequent Calculus

In Section 3.1 we introduced normal deductions which embody the strategy that proof search should proceed only bottom-up via introduction rules and top-down via elimination rules. The bi-directional nature of this calculus makes it somewhat unwieldy when it comes to the study of meta-theoretic properties and, in particular, complicates its completeness proof. In this section we develop a closely related calculus in which all proof search steps proceed bottom-up. Pictorially, we would like to flip the elimination rules upside-down.



This transformation turns introduction rules into so-called right rules, and upside-down elimination rules into so-called left rules. We have two judgments, A *left* (A is a proposition on the left) and A *right* (A is a proposition on the right). They are assembled into the form of a hypothetical judgment

$$u_1{:}A_1\ \mathit{left}, \ldots, u_n{:}A_n\ \mathit{left} \vdash A\ \mathit{right}.$$

We call such a hypothetical judgment a *sequent*.

Note that the proposition A on the right directly corresponds to the proposition whose truth is established by a natural deduction. On the other hand, propositions on the left do *not* directly correspond to hypotheses in natural deduction, since in general they include hypotheses and propositions derived from them by elimination rules.

Keeping this intuition in mind, the inference rules for sequents can now be constructed mechanically from the rules for normal and extracting derivations. To simplify the notation, we denote the sequent above by

$$A_1, \ldots, A_n \Longrightarrow A$$

where the judgments *left* and *right* are implied by the position of the propositions. Moreover, labels u_i are suppressed until we introduce proof terms. Finally, left rules may be applied to any left proposition. Since the order of the left propositions is irrelevant, we write Γ, A instead of the more pedantic Γ, A, Γ'.

Initial Sequents. These correspond to the coercion from extraction to normal derivations, and *not* to the use of hypotheses in natural deductions.

$$\frac{}{\Gamma, A \Longrightarrow A}\,\text{init}$$

Conjunction. The right and left rules are straightforward and provide a simple illustration of the translation, in particular in the way the elimination rules are turned upside-down.

$$\frac{\Gamma \Longrightarrow A \qquad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\,\wedge\text{R}$$

$$\frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\,\wedge\text{L}_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\,\wedge\text{L}_2$$

In the introduction rule (read bottom-up), we propagate Γ to both premises. This reflects that in natural deduction we can use any available assumption freely in both subdeductions. Furthermore, in the elimination rule the hypothesis $A \wedge B$ *left* persists. This reflects that assumptions in natural deduction may be used more than once. Later we analyze which of these hypotheses are actually needed and eliminate some redundant ones. For now, however, they are useful because they allow us to give a very direct translation to and from normal natural deductions.

Implication. The right rule for implication is straightforward. The left rule requires some thought. Using an extracted implication $A \supset B$ gives rise to two subgoals: we have to find a normal proof of A, but we also still have to prove our overall goal, now with the additional extracted proposition B.

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\,\supset\text{R} \qquad \frac{\Gamma, A \supset B \Longrightarrow A \qquad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\,\supset\text{L}$$

Disjunction. This introduces no new considerations.

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\,\vee\text{R}_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\,\vee\text{R}_2$$

$$\frac{\Gamma, A \vee B, A \Longrightarrow C \qquad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\,\vee\text{L}$$

Negation. Negation requires a judgment parametric in a proposition. Sometimes, this is encoded as an empty right-hand side (see Exercise 3.6).

$$\frac{\Gamma, A \Longrightarrow p}{\Gamma \Longrightarrow \neg A}\,\neg\text{R}^p \qquad \frac{\Gamma, \neg A \Longrightarrow A}{\Gamma, \neg A \Longrightarrow C}\,\neg\text{L}$$

Truth. By our general method, there is no left rule, only a right rule which models the introduction rule.

$$\frac{}{\Gamma \Longrightarrow \top}\ \top R$$

Falsehood. Again by our general method, there is no right rule, only a left rule which models the (upside-down) elimination rule.

$$\frac{}{\Gamma, \bot \Longrightarrow C}\ \bot L$$

Universal Quantification. These require only a straightforward transcription, with the appropriate translation of the side condition.

$$\frac{\Gamma \Longrightarrow [a/x]A}{\Gamma \Longrightarrow \forall x.\ A}\ \forall R^a \qquad \frac{\Gamma, \forall x.\ A, [t/x]A \Longrightarrow C}{\Gamma, \forall x.\ A \Longrightarrow C}\ \forall L$$

Existential Quantification. Again, the rules can be directly constructed from the introduction and elimination rule of natural deduction.

$$\frac{\Gamma \Longrightarrow [t/x]A}{\Gamma \Longrightarrow \exists x.\ A}\ \exists R \qquad \frac{\Gamma, \exists x.\ A, [a/x]A \Longrightarrow C}{\Gamma, \exists x.\ A \Longrightarrow C}\ \exists L^a$$

The intended theorem describing the relationship between sequent calculus and natural deduction states that $\Gamma^{\downarrow} \vdash A \Uparrow$ if and only if $\Gamma \Longrightarrow A$. *Prima facie* is unlikely that we can prove either of these directions without further generalization, since the judgments $\Gamma^{\downarrow} \vdash A \Uparrow$ and $\Gamma^{\downarrow} \vdash A \downarrow$ are mutually recursive, and the statement above does not even mention the latter.

In preparation for the upcoming proof, we recall the general property of hypothetical judgments, namely that we can substitute a derivation of the appropriate judgment for a hypothesis. When applied to normal and extracting derivations, this yields the following property.

Lemma 3.5 (Substitution Property for Extractions)

1. *If* $\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash C \Uparrow$ *and* $\Gamma_1^{\downarrow} \vdash A \downarrow$ *then* $\Gamma_1^{\downarrow}, \Gamma_2^{\downarrow} \vdash C \Uparrow$.
2. *If* $\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash C \downarrow$ *and* $\Gamma_1^{\downarrow} \vdash A \downarrow$ *then* $\Gamma_1^{\downarrow}, \Gamma_2^{\downarrow} \vdash C \downarrow$.

Proof: By induction on the structure of the given derivations of $C \Uparrow$ and $C \downarrow$. In the case where the hypothesis is used we employ weakening, that is, we adjoin the additional hypotheses $\Gamma_2^{\downarrow}$ to every judgment in the derivation of $\Gamma_1^{\downarrow} \vdash A \downarrow$. □

Using this lemma, a direct proof goes through (somewhat surprisingly).

Theorem 3.6 (Soundness of Sequent Calculus)
If $\Gamma \Longrightarrow C$ *then* $\Gamma^{\downarrow} \vdash C \Uparrow$.

Proof: By induction on the structure of the given derivation $\mathcal{S}$. We show a few representative cases.

Case: Initial sequents.

$$\frac{}{\Gamma, C \Longrightarrow C}\text{init}$$

$\Gamma^{\downarrow}, u{:}C \downarrow \vdash C \downarrow$	By hypothesis u
$\Gamma^{\downarrow}, u{:}C \downarrow \vdash C \Uparrow$	By rule $\downarrow\Uparrow$

This case confirms that initial sequents correspond to the coercion from extractions to normal deductions.

Case: Implication right rule.

$$\frac{\begin{array}{c}\mathcal{S}_2\\ \Gamma, C_1 \Longrightarrow C_2\end{array}}{\Gamma \Longrightarrow C_1 \supset C_2}\supset\text{R}$$

$\Gamma^{\downarrow}, u{:}C_1 \downarrow \vdash C_2 \Uparrow$	By i.h. on $\mathcal{S}_2$
$\Gamma^{\downarrow} \vdash C_1 \supset C_2 \Uparrow$	By rule $\supset\text{I}^u$

This case exemplifies how right rules correspond directly to introduction rules.

Case: Implication left rule.

$$\frac{\begin{array}{c}\mathcal{S}_1\\ \Gamma, A_1 \supset A_2 \Longrightarrow A_1\end{array} \qquad \begin{array}{c}\mathcal{S}_2\\ \Gamma, A_1 \supset A_2, A_2 \Longrightarrow C\end{array}}{\Gamma, A_1 \supset A_2 \Longrightarrow C}\supset\text{L}$$

$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \vdash A_1 \Uparrow$	By i.h. on $\mathcal{S}_1$
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \vdash A_1 \supset A_2 \downarrow$	By hypothesis u
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \vdash A_2 \downarrow$	By rule $\supset$E
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow, w{:}A_2 \downarrow \vdash C \Uparrow$	By i.h. on $\mathcal{S}_2$
$\Gamma^{\downarrow}, u{:}A_1 \supset A_2 \downarrow \vdash C \Uparrow$	By substitution property (Lemma 3.5)

This case illustrates how left rules correspond to elimination rules. The general pattern is that the result of applying the appropriate elimination rule is substituted for a hypothesis.

□

The proof of completeness is somewhat trickier—we first need to generalize the induction hypothesis. Generalizing a desired theorem so that a direct inductive proof is possible often requires considerable ingenuity and insight into the problem. In this particular case, the generalization is of medium difficulty.

The reader who has not seen the proof is invited to test his understanding by carrying out the generalization and proof himself before reading on.

The nature of a sequent as a hypothetical judgment gives rise to several general properties we will take advantage of. We make two of them, weakening and contraction, explicit in the following lemma.

Lemma 3.7 (Structural Properties of Sequents)

1. *(Weakening) If* $\Gamma \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$.
2. *(Contraction) If* $\Gamma, A, A \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$.

Proof: First, recall our general convention that we consider the hypotheses of a sequent modulo permutation. We prove each property by a straightforward induction over the structure of the derivation. In the case of weakening we adjoin an unused hypothesis *A left* to each sequent in the derivation. In the case of contraction we replace any use of either of the two hypotheses by a common hypothesis. □

The theorem below only establishes the completeness of sequent derivations with respect to normal deductions. That is, at this point we have not established the completeness of sequents with respect to arbitrary natural deductions which is more difficult.

Theorem 3.8 (Completeness of Sequent Derivations)

1. *If* $\Gamma^{\downarrow} \vdash C \Uparrow$ *then* $\Gamma \Longrightarrow C$.
2. *If* $\Gamma^{\downarrow} \vdash A \downarrow$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.

Proof: By induction on the structure of the given derivations $\mathcal{I}$ and $\mathcal{E}$. We show some representative cases.

Case: Use of hypotheses.

$$\mathcal{E} = \dfrac{}{\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash A \downarrow}\, u$$

$\Gamma_1, A, \Gamma_2, A \Longrightarrow C$	Assumption
$\Gamma_1, A, \Gamma_2 \Longrightarrow C$	By contraction (Lemma 3.7)

Case: Coercion.

$$\mathcal{I} = \dfrac{\begin{array}{c}\mathcal{E}\\ \Gamma^{\downarrow} \vdash C \downarrow\end{array}}{\Gamma^{\downarrow} \vdash C \Uparrow}\, {\downarrow}{\Uparrow}$$

$\Gamma, C \Longrightarrow C$	By rule init
$\Gamma \Longrightarrow C$	By i.h. on $\mathcal{E}$

Case: Implication introduction.

$$\mathcal{I} = \dfrac{\begin{matrix}\mathcal{I}_2 \\ \Gamma^{\downarrow}, u{:}C_1 \downarrow \; \vdash C_2 \Uparrow\end{matrix}}{\Gamma^{\downarrow} \vdash C_1 \supset C_2 \Uparrow} \supset\mathrm{I}^u$$

$\Gamma, C_1 \Longrightarrow C_2$	By i.h. on $\mathcal{I}_2$
$\Gamma \Longrightarrow C_1 \supset C_2$	By rule $\supset$R

Case: Implication elimination.

$$\mathcal{E} = \dfrac{\begin{matrix}\mathcal{E}_2 \\ \Gamma^{\downarrow} \vdash A_1 \supset A_2 \downarrow\end{matrix} \qquad \begin{matrix}\mathcal{I}_1 \\ \Gamma^{\downarrow} \vdash A_1 \Uparrow\end{matrix}}{\Gamma^{\downarrow} \vdash A_2 \downarrow} \supset\mathrm{E}$$

$\Gamma, A_2 \Longrightarrow C$	Assumption
$\Gamma, A_1 \supset A_2, A_2 \Longrightarrow C$	By weakening (Lemma 3.7)
$\Gamma \Longrightarrow A_1$	By i.h. on $\mathcal{I}_1$
$\Gamma, A_1 \supset A_2 \Longrightarrow A_1$	By weakening (Lemma 3.7)
$\Gamma, A_1 \supset A_2 \Longrightarrow C$	By rule $\supset$L
$\Gamma \Longrightarrow C$	By i.h. on $\mathcal{E}_2$

$\square$

In order to establish soundness and completeness with respect to arbitrary natural deductions we establish a connection to annotated natural deductions. Recall that this is an extension of normal deductions which we showed sound and complete with respect to arbitrary natural deduction in Theorems 3.2 and 3.3. We related annotated natural deductions to the sequent calculus by adding a rule called cut.

We write the extended judgment of sequent derivations with cut as $\Gamma \stackrel{+}{\Longrightarrow} C$. It is defined by copies of all the rules for $\Gamma \Longrightarrow C$, plus the rule of cut:

$$\dfrac{\Gamma \stackrel{+}{\Longrightarrow} A \qquad \Gamma, A \stackrel{+}{\Longrightarrow} C}{\Gamma \stackrel{+}{\Longrightarrow} C}\,\mathrm{cut}$$

Thought of from the perspective of bottom-up proof construction, this rule corresponds to proving and then assuming a lemma A during a derivation.

Theorem 3.9 (Soundness of Sequent Calculus with Cut)
If $\Gamma \stackrel{+}{\Longrightarrow} C$ *then* $\Gamma^{\downarrow} \vdash^{+} C \Uparrow$.

Proof: As in Theorem 3.6 by induction on the structure of the given derivation $\mathcal{S}$, with one additional case.

Case: Cut.

$$\mathcal{S} = \dfrac{\begin{array}{cc} \mathcal{S}_1 & \mathcal{S}_2 \\ \Gamma \Longrightarrow A & \Gamma, A \Longrightarrow C \end{array}}{\Gamma \Longrightarrow C}\,\text{cut}$$

$\Gamma^{\downarrow} \vdash^{+} A \Uparrow$	By i.h. on $\mathcal{S}_1$
$\Gamma^{\downarrow} \vdash^{+} A \downarrow$	By rule $\Uparrow\downarrow$
$\Gamma^{\downarrow}, u{:}A \downarrow\ \vdash^{+} C \Uparrow$	By i.h. on $\mathcal{S}_2$
$\Gamma^{\downarrow} \vdash^{+} C \Uparrow$	By substitution (Lemma 3.5, generalized)

We see that, indeed, cut corresponds to the coercion from normal to extraction derivations.

□

Theorem 3.10 (Completeness of Sequent Calculus with Cut)

1. *If* $\Gamma^{\downarrow} \vdash^{+} C \Uparrow$ *then* $\Gamma \stackrel{+}{\Longrightarrow} C$.
2. *If* $\Gamma^{\downarrow} \vdash^{+} A \downarrow$ *and* $\Gamma, A \stackrel{+}{\Longrightarrow} C$ *then* $\Gamma \stackrel{+}{\Longrightarrow} C$.

Proof: As in the proof of Theorem 3.10 with one additional case.

Case: Coercion from normal to extraction derivations.

$$\mathcal{E} = \dfrac{\begin{array}{c} \mathcal{I} \\ \Gamma^{\downarrow} \vdash^{+} A \Uparrow \end{array}}{\Gamma^{\downarrow} \vdash^{+} A \downarrow}\,\Uparrow\downarrow$$

$\Gamma \Longrightarrow A$	By i.h. on $\mathcal{I}$
$\Gamma, A \Longrightarrow C$	By assumption
$\Gamma \Longrightarrow C$	By rule cut

□

The central property of the sequent calculus is that the cut rule is redundant. That is, if $\Gamma \stackrel{+}{\Longrightarrow} C$ then $\Gamma \Longrightarrow C$. This so-called cut elimination theorem (Gentzen's *Hauptsatz* [Gen35]) is one of the central theorems of logic. As an immediately consequence we can see that not every proposition has a proof, since no rule is applicable to derive $\cdot \Longrightarrow \bot$. In the system with cut, a derivation of this sequent might end in the cut rule and consistency is not at all obvious. The proof of cut elimination and some of its many consequences are the subject of the next section.

3.4 Cut Elimination

This section is devoted to proving that the rule of cut is redundant in the sequent calculus. First we prove that cut is *admissible*: whenever the premises of the cut rule are derivable in the sequent calculus *without cut*, then the conclusion is. It is a simple observation that adding an admissible rule to a deductive system does not change the derivable judgments. Formally, this second step is an induction over the structure of a derivation that may contain cuts, proving that if $\Gamma \overset{+}{\Longrightarrow} C$ then $\Gamma \Longrightarrow C$.

There is a stronger property we might hope to prove for cut: it could be a *derived* rule of inference. Derived rules have a direct deduction of the conclusion from the premises within the given system. For example,

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B \qquad \Gamma \vdash C}{\Gamma \vdash A \wedge (B \wedge C)}$$

is a derived rule, as evidenced by the following deduction:

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Gamma \vdash B \qquad \Gamma \vdash C}{\Gamma \vdash B \wedge C}\wedge I}{\Gamma \vdash A \wedge (B \wedge C)}\wedge I.$$

Derived rules have the property that they remain valid under all extensions of a given system. Admissible rules, on the other hand, have to be reconsidered when new connectives or inference rules are added to a system, since these rules may invalidate the proof of admissibility.

It turns out that cut is only admissible, but not derivable in the sequent calculus. Therefore, we will prove the following theorem:

If $\Gamma \Longrightarrow A$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.

We call A the *cut formula*. Also, each left or right rule in the sequent calculus focuses on an occurrence of a proposition in the conclusion, called the *principal formula* of the inference.

The proof combines two ideas: induction over the structure of the cut formula with induction over the structures of the two given derivations. They are combined into one nested induction: an outer induction over the structure of the cut formula and an inner induction over the structure of the derivations of the premises. The outer induction over the structure of the cut formula is related to local reductions in natural deduction (see Exercise 3.7).

Theorem 3.11 (Admissibility of Cut)
If $\Gamma \Longrightarrow A$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.

Proof: By nested inductions on the structure of A, the derivation $\mathcal{D}$ of $\Gamma \Longrightarrow A$ and $\mathcal{E}$ of $\Gamma, A \Longrightarrow C$. More precisely, we appeal to the induction hypothesis either with a strictly smaller cut formula, or with an identical cut formula and two derivations, one of which is strictly smaller while the other stays the same. The proof is constructive, which means we show how to transform

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad\text{and}\quad \begin{array}{c}\mathcal{E}\\ \Gamma, A \Longrightarrow C\end{array} \quad\text{to}\quad \begin{array}{c}\mathcal{F}\\ \Gamma \Longrightarrow C.\end{array}$$

The proof is divided into several classes of cases. More than one case may be applicable, which means that the algorithm for constructing the derivation of $\Gamma \Longrightarrow C$ from the two given derivations is naturally non-deterministic.

Case: $\mathcal{D}$ is an initial sequent.

$$\mathcal{D} = \frac{}{\Gamma', A \Longrightarrow A}\,\text{init}$$

$\Gamma = \Gamma', A$	This case
$\Gamma', A, A \Longrightarrow C$	Derivation $\mathcal{E}$
$\Gamma', A \Longrightarrow C$	By contraction (Lemma 3.7)
$\Gamma \Longrightarrow C$	By equality

Case: $\mathcal{E}$ is an initial sequent using the cut formula.

$$\mathcal{E} = \frac{}{\Gamma, A \Longrightarrow A}\,\text{init}$$

$C = A$	This case
$\Gamma \Longrightarrow A$	Derivation $\mathcal{D}$

Case: $\mathcal{E}$ is an initial sequent not using the cut formula.

$$\mathcal{E} = \frac{}{\Gamma', C, A \Longrightarrow C}\,\text{init}$$

$\Gamma = \Gamma', C$	This case
$\Gamma', C \Longrightarrow C$	By rule init
$\Gamma \Longrightarrow C$	By equality

Case: A is the principal formula of the final inference in both $\mathcal{D}$ and $\mathcal{E}$. There are a number of subcases to consider, based on the last inference in $\mathcal{D}$ and $\mathcal{E}$. We show some of them.

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Longrightarrow A_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \wedge A_2}\,\wedge\text{R}$$

$$\text{and}\mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A_1 \wedge A_2, A_1 \Longrightarrow C\end{array}}{\Gamma, A_1 \wedge A_2 \Longrightarrow C}\,\wedge\text{L}_1$$

$\Gamma, A_1 \Longrightarrow C$ By i.h. on $A_1 \wedge A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C$ By i.h. on A_1 from above and $\mathcal{D}_1$

Actually we have ignored a detail: in the first appeal to the induction hypothesis, $\mathcal{E}_1$ has an additionaly hypothesis (A_1 *left*) and therefore does not match the statement of the theorem precisely. However, we can always weaken $\mathcal{D}$ to include this additional hypothesis without changing the structure of $\mathcal{D}$ (see the proof of Lemma 3.7) and then appeal to the induction hypothesis. We will not be explicit about these trivial weakening steps in the remaining cases.

Subcase:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma, A_1 \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \supset A_2} \supset\mathrm{R}$$

$$\text{and}\quad \mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A_1 \supset A_2 \Longrightarrow A_1\end{array} \qquad \begin{array}{c}\mathcal{E}_2\\ \Gamma, A_1 \supset A_2, A_2 \Longrightarrow C\end{array}}{\Gamma, A_1 \supset A_2 \Longrightarrow C} \supset\mathrm{L}$$

$\Gamma \Longrightarrow A_1$ By i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow A_2$ By i.h. on A_1 from above and $\mathcal{D}_2$
$\Gamma, A_2 \Longrightarrow C$ By i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C$ By i.h. on A_2 from above

Subcase:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A_1 \Longrightarrow p\end{array}}{\Gamma \Longrightarrow \neg A_1} \neg\mathrm{R}^p$$

$$\text{and}\quad \mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, \neg A_1 \Longrightarrow A_1\end{array}}{\Gamma, \neg A_1 \Longrightarrow C} \neg\mathrm{L}$$

$\Gamma \Longrightarrow A_1$ By i.h. on $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma, A_1 \Longrightarrow C$ By substitution for parameter C in $\mathcal{D}_1$
$\Gamma \Longrightarrow C$ By i.h. on A_1 from above

Note that the condition that p be a new parameter in $\mathcal{D}_1$ is necessary to guarantee that in the substitution step above we have $[C/p]A_1 = A_1$ and $[C/p]\Gamma = \Gamma$.

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \Longrightarrow [t/x]A_1\end{array}}{\Gamma \Longrightarrow \exists x.\ A_1}\ \exists\mathrm{R}$$

$$\text{and}\quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, \exists x.\ A_1, [a/x]A_1 \Longrightarrow C\end{array}}{\Gamma, \exists x.\ A_1 \Longrightarrow C}\ \exists\mathrm{L}^a$$

$\Gamma, [t/x]A_1 \Longrightarrow C$	By substitution for parameter a in $\mathcal{E}_1$
$\Gamma, [t/x]A_1 \Longrightarrow C$	By i.h. on $\exists x.\ A_1$, $\mathcal{D}$ and $[t/a]\mathcal{E}_1$
$\Gamma \Longrightarrow C$	By i.h. on $[t/x]A_1$ from $\mathcal{D}_1$ and above

Note that this case requires that $[t/x]A_1$ is considered smaller than $\exists x.\ A_1$. Formally, this can be justified by counting the number of quantifiers and connectives in a proposition and noting that the term t does not contain any. A similar remark applies to check that $[t/a]\mathcal{E}_1$ is smaller than $\mathcal{E}$. Also note how the side condition that a must be a new parameter in the $\exists$L rule is required in the substitution step to conclude that $[t/a]\Gamma = \Gamma$, $[t/a][a/x]A_1 = [t/x]A_1$, and $[t/a]C$.

Case: A is not the principal formula of the last inference in $\mathcal{D}$. In that case $\mathcal{D}$ must end in a left rule and we can appeal to the induction hypothesis on one of its premises. We show some of the subcases.

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma', B_1 \wedge B_2, B_1 \Longrightarrow A\end{array}}{\Gamma', B_1 \wedge B_2 \Longrightarrow A}\ \wedge\mathrm{L}_1$$

$\Gamma = \Gamma', B_1 \wedge B_2$	This case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	By i.h. on A, $\mathcal{D}_1$ and $\mathcal{E}$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	By rule $\wedge\mathrm{L}_1$
$\Gamma \Longrightarrow C$	By equality

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma', B_1 \supset B_2 \Longrightarrow B_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma', B_1 \supset B_2, B_2 \Longrightarrow A\end{array}}{\Gamma', B_1 \supset B_2 \Longrightarrow A}\ \supset\mathrm{L}$$

$\Gamma = \Gamma', B_1 \supset B_2$	This case
$\Gamma', B_1 \supset B_2, B_2 \Longrightarrow C$	By i.h. on A, $\mathcal{D}_2$ and $\mathcal{E}$
$\Gamma', B_2 \supset B_2 \Longrightarrow C$	By rule $\supset$L on $\mathcal{D}_1$ and above
$\Gamma \Longrightarrow C$	By equality

Case: A is not the principal formula of the last inference in $\mathcal{E}$. This overlaps with the previous case, since A may not be principal on either side. In this case, we appeal to the induction hypothesis on the subderivations of $\mathcal{E}$ and directly infer the conclusion from the results. We show some of the subcases.

Subcase:

$$\mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, A \Longrightarrow C_1\end{array} \qquad \begin{array}{c}\mathcal{E}_2 \\ \Gamma, A \Longrightarrow C_2\end{array}}{\Gamma, A \Longrightarrow C_1 \wedge C_2} \wedge\mathrm{R}$$

$C = C_1 \wedge C_2$	This case
$\Gamma \Longrightarrow C_1$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C_2$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C_1 \wedge C_2$	By rule $\wedge$R on above

Subcase:

$$\mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma', B_1 \wedge B_2, B_1, A \Longrightarrow C\end{array}}{\Gamma', B_1 \wedge B_1, A \Longrightarrow C} \wedge\mathrm{L}_1$$

$\Gamma = \Gamma', B_1 \wedge B_2$	This case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	By rule $\wedge\mathrm{L}_1$ from above

□

As mentioned above, it is a general property of deductive system that adding an admissible rule does not change the derivable judgments. We show the argument in this special case.

Theorem 3.12 (Cut Elimination)
If $\Gamma \overset{+}{\Longrightarrow} C$ *then* $\Gamma \Longrightarrow C$.

Proof: In each case except cut we simply appeal to the induction hypotheses and reapply the same rule on the resulting cut-free derivations. So we write out only the case of cut.

Case:

$$\mathcal{D}^+ = \dfrac{\begin{array}{c}\mathcal{D}_1^+ \\ \Gamma \overset{+}{\Longrightarrow} A\end{array} \qquad \begin{array}{c}\mathcal{D}_2^+ \\ \Gamma, A \overset{+}{\Longrightarrow} C\end{array}}{\Gamma \overset{+}{\Longrightarrow} C} \mathrm{cut}$$

$\Gamma \Longrightarrow A$	By i.h. on $\mathcal{D}_1^+$
$\Gamma, A \Longrightarrow C$	By i.h. on $\mathcal{D}_2^+$
$\Gamma \Longrightarrow C$	By admissibility of cut (Theorem 3.11)

□

3.5 Applications of Cut Elimination

The cut elimination theorem is the final piece needed to complete our study of natural deduction and normal natural deduction and at the same time the springboard to the development of efficient theorem proving procedures. Our proof in the previous section is constructive and therefore contains an algorithm for cut elimination. Because the cases are not mutually exclusive, the algorithm is non-deterministic. However, the resulting derivation should always be the same. While this property does not quite hold, the different derivations can be shown to be equivalent in a natural sense. This is called the *confluence* property for intuitionistic cut elimination modulo commutative conversions. It it is not implicit in our proof, but has to be established separately. On the other hand, our proof shows that any possible execution of the cut-elimination algorithm terminates. This is called the *strong normalization* property for the sequent calculus.

By putting the major results of this chapter together we can now prove the normalization theorem for natural deduction.

Theorem 3.13 (Normalization for Natural Deduction)
If $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash A \Uparrow$.

Proof: Direct from previous theorems.

$\Gamma \vdash A$	Assumption
$\Gamma^{\downarrow} \vdash^{+} A \Uparrow$	By completeness of annotated deductions (Theorem 3.3)
$\Gamma \stackrel{+}{\Longrightarrow} A$	By completeness of sequent calculus with cut (Theorem 3.10)
$\Gamma \Longrightarrow A$	By cut elimination (Theorem 3.12)
$\Gamma^{\downarrow} \vdash A \Uparrow$	By soundness of sequent calculus (Theorem 3.6)

□

Among the other consequences of cut elimination are consistency and various independence results.

Corollary 3.14 (Consistency) *There is no deduction of* $\vdash \bot$.

Proof: Assume there is a deduction $\vdash \bot$. By the results of this chapter then $\cdot \Longrightarrow \bot$. However, this sequent cannot be the conclusion of any inference rule in the (cut-free) sequent calculus. Therefore $\vdash \bot$ cannot be derivable. □

In the same category are the following two properties. As in the proof above, we analyze the inference rules which may have led to a given conclusion. This proof technique is called *inversion*.

Corollary 3.15 (Disjunction and Existential Property)

1. *If* $\vdash A \vee B$ *then either* $\vdash A$ *or* $\vdash B$.
2. *If* $\vdash \exists x.\ A$ *then* $\vdash [t/x]A$ *for some* t.

Proof: Direct by inversion on possible sequent derivations in both cases.

1. Assume $\vdash A \lor B$. Then $\cdot \Longrightarrow A \lor B$. By inversion, either $\cdot \Longrightarrow A$ or $\cdot \Longrightarrow B$. Therefore $\vdash A$ or $\vdash B$.

2. Assume $\exists x.\ A$. then $\cdot \Longrightarrow \exists x.\ A$. By inversion, $\cdot \Longrightarrow [t/x]A$ for some t. Hence $\vdash [t/x]A$.

□

Note that the disjunction and existential properties rely on a judgment without hypotheses. For example, we have $B \lor A \Longrightarrow A \lor B$, but neither $B \lor A \Longrightarrow A$ for $B \lor A \Longrightarrow B$ hold.

The second class of properties are *independence* results which demonstrate that certain judgments are not derivable. As a rule, these are parametric judgments some instances of which may be derivable. For example, we will show that the law of excluded middle is independent. Nonetheless, there are some propositions A for which we can show $\vdash A \lor \neg A$ (for example, take $A = \bot$).

Corollary 3.16 (Independence of Excluded Middle)
There is no deduction of $\vdash A \lor \neg A$ for arbitrary A.

Proof: Assume there is a deduction of $\vdash A \lor \neg A$. By the result of this section then $\cdot \Longrightarrow A \lor \neg A$. By inversion now either $\cdot \Longrightarrow A$ or $\cdot \Longrightarrow \neg A$. The former judgment (which is parametric in A) has no derivation. By inversion, the latter can only be infered from $A \Longrightarrow p$ for a new parameter p. But there is no inference rule with this conclusion, and hence there cannot be a deduction of $\vdash A \lor \neg A$. □

3.6 Proof Terms for Sequent Derivations

In this section we address the question of how to assign proof terms to sequent calculus derivations. There are essentially two possibilities: we can either develop a new proof term calculus specifically for sequent derivations, or we can directly assign natural deduction proof terms. The former approach can be found, for example, in [Pfe95]. The latter is more appropriate for our purposes here, since we view natural deductions as defining truth and since we already devised methods for compact representations in Section 3.2.

We define a new judgment, $\Gamma \Longrightarrow I : A$, maintaining that $\Gamma \vdash I : A$. For this purpose we abandon the previous convention of omitting labels for hypotheses, since proof terms need to refer to them. On the other hand, we still consider assumptions modulo permutations in order to simplify notation. We use the compact proof terms here only for simplicity.

The proof terms to be assigned to each inference rule can be determined by a close examination of the soundness proof for the sequent calculus (Theorem 3.6). Since that proof is constructive, it contains an algorithm for translating a sequent derivation to a normal natural deduction. We just have to write down the corresponding proof terms.

Initial Sequents. These are straightforward.

$$\frac{}{\Gamma, u{:}A \Longrightarrow u : A}\,\text{init}$$

Note that there may be several hypotheses A with different labels. In the shorthand notation without labels before, it is ambiguous which one was used.

Conjunction. The right rule is straightforward, since it is isomorphic to the introduction rule for natural deduction. The left rules require a substitution to be carried out, just as in the proof of Theorem 3.6.

$$\frac{\Gamma \Longrightarrow I : A \qquad \Gamma \Longrightarrow J : B}{\Gamma \Longrightarrow \langle I, J\rangle : A \wedge B}\,\wedge\text{R}$$

$$\frac{\Gamma, u{:}A \wedge B, w{:}A \Longrightarrow I : C}{\Gamma, u{:}A \wedge B \Longrightarrow [\mathrm{fst}\, u/w]I : C}\,\wedge\text{L}_1 \qquad \frac{\Gamma, u{:}A \wedge B, w{:}B \Longrightarrow I : C}{\Gamma, u{:}A \wedge B \Longrightarrow [\mathrm{snd}\, u/w]I : C}\,\wedge\text{L}_2$$

There are two potential efficiency problems in the proof term assignment for the left rule. The first is that if w is used many times in I, then $\mathrm{fst}\, u$ or $\mathrm{snd}\, u$ may be replicated many times, leading to a large proof. The second is that when a number of successive left rules are encountered, the term I we substitute into will be traversed many times. These problems can be avoided in several ways (see Exercise **??**).

Implication. The pattern of the previous right and left rules continues here.

$$\frac{\Gamma, u{:}A \Longrightarrow I : B}{\Gamma \Longrightarrow \lambda u.\, I : A \supset B}\,\supset\text{R}$$

$$\frac{\Gamma, u{:}A \supset B \Longrightarrow J : A \qquad \Gamma, u{:}A \supset B, w{:}B \Longrightarrow I : C}{\Gamma, u{:}A \supset B \Longrightarrow [u\, J/w]I : C}\,\supset\text{L}$$

Disjunction. This introduces no new considerations.

$$\frac{\Gamma \Longrightarrow I : A}{\Gamma \Longrightarrow \mathrm{inl}\, I : A \vee B}\,\vee\text{R}_1 \qquad \frac{\Gamma \Longrightarrow J : B}{\Gamma \Longrightarrow \mathrm{inr}\, J : A \vee B}\,\vee\text{R}_2$$

$$\frac{\Gamma, u{:}A \vee B, v{:}A \Longrightarrow I : C \qquad \Gamma, u{:}A \vee B, w{:}B \Longrightarrow J : C}{\Gamma, u{:}A \vee B \Longrightarrow (\,\textbf{case}\ u\ \textbf{of}\ \mathrm{inl}\, v \Rightarrow I \mid \mathrm{inr}\, w \Rightarrow J) : C}\,\vee\text{L}$$

Negation. This is similar to implication.[1]

$$\frac{\Gamma, u{:}A \Longrightarrow I : p}{\Gamma \Longrightarrow \mu^p u.\, I : \neg A}\,\neg\text{R}^p \qquad \frac{\Gamma, u{:}\neg A \Longrightarrow I : A}{\Gamma, u{:}\neg A \Longrightarrow u \cdot I : C}\,\neg\text{L}$$

[1] [*add to compact proof term section?*]

Truth. This is trivial, since there is no left rule.

$$\frac{}{\Gamma \Longrightarrow \langle\,\rangle : \top}\,\top R$$

Falsehood. Again, this is immediate.

$$\frac{}{\Gamma, u{:}\bot \Longrightarrow \mathrm{abort} u : C}\,\bot L$$

To treat the quantifiers we extend our proof term calculus to handle the quantifier rules. We overload the notation by reusing λ-abstraction and pairing. There is no ambiguity, because the proof term for universal quantification binds a term variable x (rather than a proof variable u), and the first component of the pair for existential quantification is a first-order term, rather than a proof term as for conjunction.

First, we show the assignment of these terms to natural deductions, then to the sequent calculus.

Universal Quantification. The proof term for a universal quantifier $\forall x.\ A$ is a function from a term t to a proof of $[t/x]A$. The elimination term applies this function.

$$\frac{\Gamma \vdash [a/x]M : [a/x]A}{\Gamma \vdash \lambda x.\ M : \forall x.\ A}\,\forall I^{a}$$

$$\frac{\Gamma \vdash M : \forall x.\ A}{\Gamma \vdash M\, t : [t/x]A}\,\forall E$$

The local reductions and expansions just mirror the corresponding operations on natural deductions.

$$\begin{array}{rcll} (\lambda x.\ M)\, t & \longrightarrow_R & [t/x]M & \\ M : \forall x.\ A & \longrightarrow_E & \lambda x.\ M\, x & (x \text{ not free in } M) \end{array}$$

Existential Quantification. The proof term for an existential $\exists x.\ A$ is a pair consisting of a witness term t and the proof of $[t/x]A$.

$$\frac{\Gamma \vdash M : [t/x]A}{\Gamma \vdash \langle t, M\rangle : \exists x.\ A}\,\exists I$$

$$\frac{\Gamma \vdash M : \exists x.\ A \qquad \Gamma, u{:}[a/x]A \vdash [a/x]N : C}{\Gamma \vdash \mathbf{let}\ \langle x, u\rangle = M\ \mathbf{in}\ N : C}\,\exists E^{a,u}$$

The local reduction for the existential quantifier has to perform two substitutions, just as on natural deductions.

$$\begin{array}{rcl} \mathbf{let}\ \langle x, u\rangle = \langle t, M\rangle\ \mathbf{in}\ N & \longrightarrow_R & [M/u][t/x]N \\ M : \exists x.\ A & \longrightarrow_E & \mathbf{let}\ \langle x, u\rangle = M\ \mathbf{in}\ \langle x, u\rangle \end{array}$$

It is once again easy to see how to divide the proof terms into introduction and elimination forms. We only show the resulting definition of compact proof terms.

$$\begin{array}{lcll}
\text{Intro Terms} \quad I & ::= & \ldots & \\
 & & \mid \lambda x.\, I & \text{Universal Quantification} \\
 & & \mid \langle t, I\rangle & \text{Existential Quantification} \\
 & & \mid \mathbf{let}\ \langle x, u\rangle = E\ \mathbf{in}\ I & \\
\text{Elim Terms} \quad E & ::= & \ldots \mid E\, t & \text{Universal Quantification}
\end{array}$$

On sequent calculus derivations, we follow the same strategy as in the preceding propositional rules.

Universal Quantification.

$$\frac{\Gamma \Longrightarrow [a/x]I : [a/x]A}{\Gamma \Longrightarrow \lambda x.\, I : \forall x.\, A}\,\forall\mathrm{R}^a \qquad \frac{\Gamma, u{:}\forall x.\, A, w{:}[t/x]A \Longrightarrow I : C}{\Gamma, u{:}\forall x.\, A \Longrightarrow [u\, t/w]I : C}\,\forall\mathrm{L}$$

Existential Quantification.

$$\frac{\Gamma \Longrightarrow I : [t/x]A}{\Gamma \Longrightarrow \langle t, I\rangle : \exists x.\, A}\,\exists\mathrm{R} \qquad \frac{\Gamma, u{:}\exists x.\, A, w{:}[a/x]A \Longrightarrow [a/x]I : C}{\Gamma, u{:}\exists x.\, A \Longrightarrow (\mathbf{let}\ \langle x, w\rangle = u\ \mathbf{in}\ I) : C}\,\exists\mathrm{L}^a$$

3.7 Exercises

Exercise 3.1 Consider a system of normal deduction where the elimination rules for disjunction and existential are allowed to end in an extraction judgment.

$$\frac{\Gamma^\downarrow \vdash A \vee B \downarrow \qquad \Gamma^\downarrow, u{:}A\downarrow \vdash C\downarrow \qquad \Gamma^\downarrow, w{:}B\downarrow \vdash C\downarrow}{\Gamma^\downarrow \vdash C\downarrow}\,\vee\mathrm{E}^{u,w}$$

$$\frac{\Gamma^\downarrow \vdash \exists x.\, A\downarrow \qquad \Gamma^\downarrow, u{:}[a/x]A\downarrow \vdash C\downarrow}{\Gamma^\downarrow \vdash C\downarrow}\,\exists\mathrm{E}^{a,u}$$

Discuss the relative merits of allowing or disallowing these rules and show how they impact the subsequent development in this Chapter (in particular, bidirectional type-checking and the relationship to the sequent calculus).

Exercise 3.2

1. Give an example of a natural deduction which is *not* normal (in the sense defined in Section 3.1), yet contains no subderivation which can be locally reduced.

2. Generalizing from the example, devise additional rules of reduction so that any natural deduction which is not normal can be reduced. You should introduce no more and no fewer rules than you need for this purpose.

3. Prove that your rules satisfy the specification in part (2).

Exercise 3.3 Write out the rules defining the judgments $\Gamma^{\downarrow} \vdash^{+} I : A \Uparrow$ and $\Gamma^{\downarrow} \vdash^{+} E : A \downarrow$ and prove Theorem 3.4. Make sure to carefully state the induction hypothesis (if it is different from the statement of the theorem) and consider all the cases.

Exercise 3.4 Fill in the missing subcases in the proof of the admissibility of cut (Theorem 3.11) where A is the principal formula in both $\mathcal{D}$ and $\mathcal{E}$.

Exercise 3.5 Consider an extension of intuitionistic logic by a universal quantifier over propositions, written as $\forall^2 p.\ A$, where p is variable ranging over propositions.

1. Show introduction and elimination rules for $\forall^2$.

2. Extend the calculus of normal and extraction derivations.

3. Show left and right rules of the sequent calulus for $\forall^2$.

4. Extend the proofs of soundness and completeness for the sequent calculus and sequent calculus with cut to accomodate the new rules.

5. Point out why the proof for admissibility of cut does not extend to this logic.

Exercise 3.6 Gentzen's original formulation of the sequent calculus for intuitionistic logic permitted the right-hand side to be empty. The introduction rule for negation then has the form

$$\frac{\Gamma, A \Longrightarrow}{\Gamma \Longrightarrow \neg A}\,\neg\mathrm{R}.$$

Write down the corresponding left rule and detail the changes in the proof for admissibility of cut. Can you explain sequents with empty right-hand sides as judgments?

Exercise 3.7 The algorithm for cut elimination implicit in the proof for admissibility of cut can be described as a set of reduction rules on sequent derivations containing cut.

1. Write out all reduction rules on the fragment containing only implication.

2. Show the extracted proof term before and after each reduction.

3. If possible, formulate a strategy of reduction on proof terms for natural deduction which directly models cut elimination under our translation.

4. Either formulate and prove a theorem about the connection of the strategies for cut elimination and reduction, or show by example why such a connection is difficult or impossible.

Exercise 3.8

1. Prove that we can restrict initial sequents in the sequent calculus to have the form $\Gamma, P \Longrightarrow P$ where P is an atomic proposition without losing completeness.
2. Determine the corresponding restriction in normal and extraction derivations and prove that they preserve completeness.
3. If you see a relationship between these properties and local reductions or expansions, explain. If you can cast it in the form of a theorem, do so and prove it.

Exercise 3.9 For each of the following propositions, prove that they are derivable in classical logic using the law of excluded middle. Furthermore, prove that they are not true in intuitionistic logic for arbitrary A, B, and C.

1. $((A \supset B) \supset A) \supset A$.
2. Any entailment in Exercise 2.8 which is only classically, but not intuitionistically true.