Estimating Interpolation Error:
A Combinatorial Approach

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Abstract

New estimates are established for the error between a function and its linear interpolant over a triangular domain. Results previously established using compactness arguments are established here using combinatorial arguments which provide explicit estimates of the constants. New results include new path embeddings for "convex" graphs and bounds on the Rayleigh quotient of complete bipartite graphs. These results are applicable to anisotropic finite element meshing.

1 Introduction

In many areas of Computer Science one approximates a more complicated function with a simpler function. Sometimes the more complicated function is given explicitly such as modeling an object in graphics. For other applications, the function may only be known implicitly, say, as the solution to a partial differential equation.

For example, in Figure 1 we approximate a quadratic polynomial in two variables by a piecewise linear function. For each of the triangles of the square we pick the piecewise linear function which agrees with the polynomial at the vertices. In the case when angles are allowed to approach 180^o degrees the approximation is not very good. Although the L^2 error (see below for definition) for the mesh using obtuse triangles is not that bad, the derivative of the error is large. This error, the H^1 error, is also known as the energy norm. Having large H^1 error is bad for several reasons. First, in graphic applications this will give solutions with very different texture than the target function. Second, the finite element method attempts to minimize the energy norm at the expense of the L^2 norm. Thus, using finite element methods may generate solutions with large L^2 error.

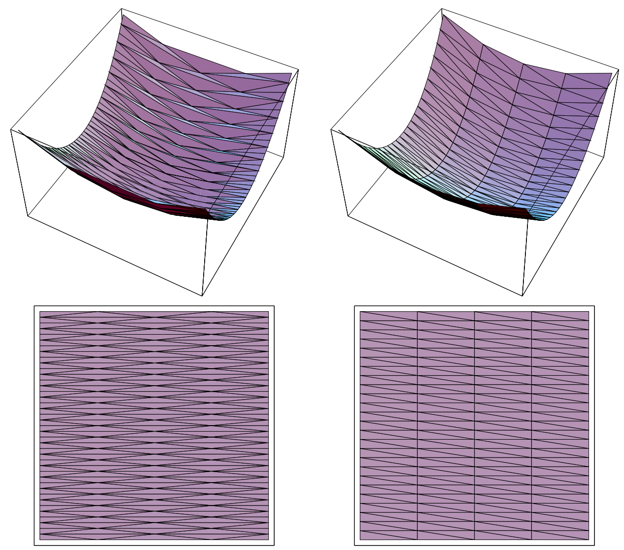

Figure 1: Approximating a Hammock with Large Angle Triangles

The goal of this paper is to understand which shaped triangles can be used in the mesh so that one can guarantee that the interpolation error is small. The general question of how to guarantee small interpolation error in higher dimensions or for quadrilaterals in 2D still has many open problems and we hope that the ideas here can be used to understand these more general cases.

In the rest of the paper we restrict our attention to the problem of bounding the interpolation error when one tries to approximate one function by another. The simplest case is when we interpolate a function $u : \Re^2 \rightarrow \Re$ by a piecewise linear function. Throughout this paper we will assume that u is defined on some domain Ω and all its second partial derivatives are square integrable on Ω. A standard way to specify a class of piecewise linear functions is by triangulating the domain Ω and considering all continuous functions over Ω which are linear on each triangle. Let u_h be such a piecewise linear function.

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There are various different measures or norms of the error, $e = u - u_h$. We will consider the following which arise in the finite element method:

- L^2 norm: $||e||_2 \equiv \int_\Omega e^2$
- H^1 semi–norm: $|e|_{H^1} \equiv \int_\Omega |De|^2 \equiv \int_\Omega (e_x^2 + e_y^2)$
- H^2 semi–norm: $|e|_{H^2} \equiv \int_\Omega |D^2 e|^2 \equiv \int_\Omega (e_{xx}^2 + 2e_{xy}^2 + e_{yy}^2)$

where e_x, e_y, e_{xx}, e_{xy}, and e_{yy} are the first and second partial derivatives of e. Analogous with Taylor approximants, estimates of the error in one norm involve bounds of u in a norm having higher derivatives. In particular, we will show that if u is bounded in the H^2 norm then for "nice" triangulations the error will be small in the H^1 norm. Let $M = (V, E, T)$ be a triangulation of the domain, where V, E, and T are the vertices, edges, and triangles of M, respectively. The simplest linear interpolant is the function u_h which agrees with u on the vertices V and is otherwise linear on the triangles, T. The ideas presented will extend to other interpolants but we will not consider them here.

To help recall these measures we first present a simple example of a triangle and a function defined on the triangle where linear interpolation over the triangle has large error as defined above. Let T be the triangle with corners $(1, 0)$, $(0, \epsilon)$, and $(-1, 0)$ for some small $\epsilon > 0$ (see Figure 1). Consider the polynomial $f(x, y) = 1 - x^2 - y/\epsilon$ which is zero at the three corners of T. If we interpolate f over T with the identically zero function so that the interpolant agrees with f at the three corners then the error is f. Thus, the partial derivatives of the error are $f_x = -2x$, $f_y = -1/\epsilon$, $f_{xx} = -2$, and $f_{yy} = f_{xy} = 0$. Using these partial derivatives we can compute the semi-norms. We get $|f|_{H^1} = \frac{2}{3}\epsilon + 1/\epsilon > 1/\epsilon$ and $|f|_{H^2} = 4\epsilon$. Thus $\frac{|f|_{H^2}}{|f|_{H^1}} < 4\epsilon^2$. Therefore as ϵ goes to zero or equivalently the largest angle goes 180^o the H_1 error is not bounded by a constant in the H^2 norm of f.

The main goal of this paper is to show that the ratio $\frac{|f|_{H^2}}{|f|_{H^1}}$ is bounded away from zero for any triangle and any function where the largest angle is bounded away from 180^o. Note that that in the above example f_x is small. We will show that the norm of any partial parallel to a side is "small".

The most important paper in the area is by Babuška and Aziz [BA76]. They showed that right triangles in two dimensions, and their analog in three dimensions, will exhibit optimal interpolation even if they have arbitrarily small volume. In two dimensions this leads to the very simple geometric requirement that "the maximum angle within any triangle be bounded away from π"; however, no simple geometric restriction is

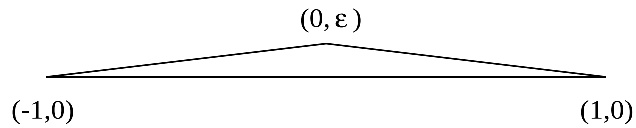

Figure 2: Large Interpolation Error for Function $f(x, y) = 1 - x^2 - y/\epsilon$ over a Triangle

known for tetrahedra. Their proof uses compactness arguments and thus no explicit bounds are known for the error.

In this paper we will show how ideas from graph embeddings can be used to explicitly estimate the errors [Lei92]. There is a fairly large amount of research in estimating eigenvalues of graphs and much of it uses graph embeddings [SJ89, GLM97]. There are several different ways of obtaining a matrix from a graph which give slightly different eigenvalues. In this paper we will use the Laplacian of a graph.

Definition 1.1. *Let $G = (V, E)$ be an edge weighted undirected graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge weights w_{ij}. The Laplacian of G is an n by n matrix B where:*

$$b_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \text{ and } (v_i, v_j) \in E \\ 0 & \text{if } i \neq j \text{ and } (v_i, v_j) \notin E \\ \sum_{k=1}^{n} |w_{ik}| & \text{if } i = j \end{cases}$$

If G is an unweighted graph then $w_{ij} = 1$ if there is an edge from v_i to v_j.

In Section 2 we show how the reformulate the problem as a discrete problem. In Sections 3 and 4 we solve the two main combinatorial problems needed to get the inequality, namely, bounding congestion for path embedding for triangles and bounding Rayleigh quotients for complete bipartite graphs. Then Section 5 reviews classical finite element interpolation theory and how the these classic methods differ from our more combinatorial methods. The main theorem is also stated this section. Finally, we list several open questions and directions for future work.

2 Interpolation Estimates for Triangles via Graph Embeddings

Throughout this section we consider a triangle T having diameter $h(T)$ and one side (the base) along the x–axis, and we show that there is a constant independent of T such that any smooth function u that vanishes at the vertices satisfies

$$\int_T u_x^2 \leq Ch(T)^2 \int_T (u_{xx}^2 + u_{xy}^2)$$

where the constant C is independent of the geometry of T. Notice that the right hand integrand is $|Du_x|^2$

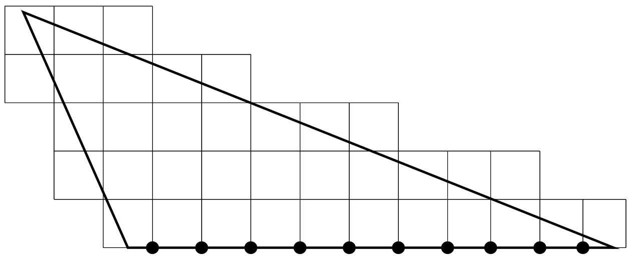

Figure 3: Meshing a Triangle

which is bounded above by $|D^2u|^2$ so that this inequality will establish Theorem 5.1. We begin by defining the function $w = u_x$, and observe that u vanishing at the vertices implies that the average value of w along the base is zero. It then suffices to prove the following lemma.

LEMMA 2.1. *Let T be a triangle with base $[0, \ell]$ on the x–axis, and let $w \in H^1(T)$ satisfy*

$$\int_0^\ell w(x,0)\,dx = 0, \tag{2.1}$$

then there is a constant C independent of T such that

$$\int_T w^2 \le Ch(T)^2 \int_T |Dw|^2,$$

where $h(T)$ is the diameter of T.

Proof: Consider the Rayleigh quotient associated with this problem.

$$I(w) = \frac{\int_T |Dw|^2}{\int_T w^2}.$$

It suffices to show that this quotient is bounded below by $c/h(T)^2$ for some constant c.

We approximate the integrals appearing in the Rayleigh quotient by finite differences. Specifically, consider a uniform rectilinear mesh having mesh size h an integral divisor of the base length ℓ, and locate the mesh with respect to the triangle by requiring the point $(h/2, 0)$ to be a grid point. We denote by $\mathcal{M}_h$ that portion of the mesh consisting of those grid points and edges that meet the triangle T (see Figure 3).

Since any function $w \in H^1(T)$ may be approximated by the restriction to T of smooth functions defined on the whole plane, we may estimate the infimum of the Rayleigh quotient by considering the infimum of $I(\cdot)$ over smooth functions satisfying the constraint (2.1). The integrals of smooth functions can be estimated using finite difference approximations. Specifically, for the mesh $\mathcal{M}_h$ we define $w_i = w(x_i)$ for mesh points x_i not on the base, and $w_i = [u(x_i + h/2, 0) - u(x_i - h/2, 0)]/h$ for mesh points x_i on the base $[0, \ell]$. These interpolations of w satisfy

$$\int_T w^2 = \lim_{h\to 0} h^2 \sum_{i\in V(\mathcal{M})} w_i^2$$

and

$$\int_T |Dw|^2 = \lim_{h\to 0} \sum_{(i,j)\in E(\mathcal{M})} (w_i - w_j)^2,$$

and $\sum_{i\in V(\mathcal{M})\cap[0,\ell]} w_i = 0$.

It follows that the Rayleigh quotient $I(w)$ can be estimated by $I(w) = \lim_{h\to 0} I_h(w)$ where

$$I_h(w) = \frac{\sum_{(i,j)\in E(\mathcal{M})}(w_i - w_j)^2}{h^2 \sum_{i\in V(\mathcal{M})} w_i^2} = \frac{\mathbf{w}^T A_h \mathbf{w}}{h^2 \mathbf{w}^T \mathbf{w}}.$$

Here A_h is the Laplacian matrix of the mesh $\mathcal{M}_h$ and $\mathbf{w}$ is the vector of interpolated w–values $\mathbf{w} = (\ldots, w_i, \ldots)$. The estimates for Rayleigh quotients of graphs that will be established in Theorem 4.2 complete the proof of the theorem. □

3 Finding Path Embeddings for Convex Bodies

In this section we show how to find a path from each internal node to each boundary node on the base for a mesh graph such as the one shown in Figure 3. We shall bound the congestion, the number of paths requesting to use an edge, and the dilation, the longest path constructed. The congestion times the dilation will be used in Section 4 to estimate interpolation error. It will be slightly easier to find for each path a collection of fractional paths whose sum is one. These less restrictive paths will work just as well in Section 4 to achieve our bound. We define our fractional path embedding below.

DEFINITION 3.1. *A* **fractional path embedding** *of a graph $G = (V, E)$ to a graph $H = (V', E')$ is a pair of mappings (1) $\phi : V \to V'$ which is one-to-one and onto; and (2) a function f from E to collection of weighted paths $\mathcal{P}$ in H, such that each path $P \in \mathcal{P} = f(v, w)$ is a weighted path in H from $\phi(v)$ to $\phi(w)$ and the sum of the weights is one. The* **congestion** *of an edge e in H is the weighted sum of the edges in the range of f which use e, and the the* **dilation** *is the number of edges in the longest path in the range of f.*

Historically, G is called the guest graph and H is called the host. An important special case has been when G is the complete graph and ϕ is the identity function. In this paper G will be a complete bipartite graph. We first start by defining a class of host graphs.

Let $\mathcal{C}$ be a compact convex body in $\mathbb{R}^2$. We shall construct a rectilinear mesh for $\mathcal{C}$. Consider the infinite rectilinear mesh $\mathcal{M}_\infty = (V, E, SQ)$ of $\mathbb{R}^2$ with unit

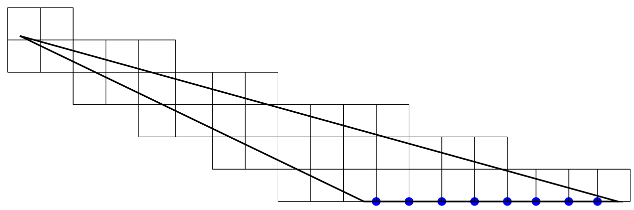

Figure 4: A General Triangle and its Mesh

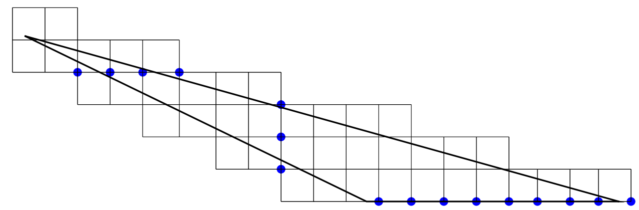

Figure 5: The Triangle Mesh from the Last Figure Decomposed into Three Right-Triangle Meshes.

mesh size, i.e., V is the set of integral points in $\mathbb{R}^2$. We define the **intersection** of $\mathcal{M}_\infty$ and $\mathcal{C}$, denoted $\mathcal{M}_\infty \cap \mathcal{C}$, to be the mesh formed from all the squares in $\mathcal{M}_\infty$ whose interiors intersect $\mathcal{C}$. For each square we include all its edges and vertices in $\mathcal{M}_\infty \cap \mathcal{C}$. We shall call any mesh constructed as above a **convex rectilinear mesh**. A **boundary node** is any vertex that is contained in fewer than four squares. We shall be interested in the meshes we get as we let the size of a fixed aspect ratio triangle T get arbitrarily large.

Let T be a triangle which is aligned with the x-axis, i.e., the base of T is contained in the x-axis, and let $\mathcal{M}_T$ be a convex rectilinear mesh of T on $n+b$ nodes. Further, let B be the distinguished subset of boundary nodes of $\mathcal{M}_T$ which also belong to the base of T (e.g. the bold nodes in Figure 4) of size b. Recall that the complete bipartite graph, $K_{n,b}$, has $n+b$ vertices and all possible edges between the first n and last b of them. We let ϕ be the natural map of the vertices of $K_{n,b}$ to those of $\mathcal{M}_T$. When we refer to a **path embedding of** $\mathcal{M}_T$ **to** B we mean a path embedding of $K_{n,b}$ onto $\mathcal{M}_T$ that maps the nodes in the natural fashion determined by ϕ.

A **right-triangle mesh** will be a mesh of a right triangle which is aligned with both the x and y-axis.

Theorem 3.1. *Let $\mathcal{M}_T$ be a convex rectilinear mesh of a triangle T and B the boundary nodes of $\mathcal{M}_T$ belonging to the base T, then there exists a fractional path embedding from $\mathcal{M}_T$ to B with congestion $O(dia \cdot b)$ and dilation $O(dia)$, where dia is the graph diameter of $\mathcal{M}_T$ and b is the size of B.*

3.1 Proof of Theorem 3.1

The proof will be in three parts: (1) First, we decompose any triangle mesh into a collection of right-triangle meshes. (2) Second, we show several results on routing in these right-triangle meshes. (3) Third, we use induction on the number of right-triangle meshes in the decomposition of our triangle to prove the result.

We consider two cases separately dependent on whether or not the opposite vertex of the triangle sits above the base or not. In the case when the opposite vertex is above the base, the mesh of the triangle can be decomposed into two right-triangle meshes and the results which we show below will suffice to handle this case. The harder case is when the oppose vertex does not lie above the base as in Figure 4. In this case we take the natural greedy decomposition of the mesh. Up to rotations and reflections we may assume that the opposite vertex is to the left of the base. We repeatedly remove the largest right-triangle from the right. The rightmost right-triangle mesh will have as its base all squares from the base of $\mathcal{M}_T$. In Figure 5 we have decomposed the mesh in Figure 4 into three right-triangle meshes. We have highlighted the nodes that are shared by these right-triangle meshes. Note that some of the nodes in the corner of these meshes are not shared between meshes and thus cannot be used in the routing. This will make the statements and the proofs for right-triangle meshes slightly more complicated. We have also highlighted the rightmost base node to make the inductive hypothesis work.

Suppose that T is a right triangle with a base of size b and a side of size a and $\mathcal{M}$ is its right triangle mesh. We list a few simple facts about $\mathcal{M}$.

1. The numbers of squares on the base and left side are $\bar{b} = \lceil b \rceil$ and $\bar{a} = \lceil a \rceil$ respectively.
2. The number of nodes is at most $ab/2 + 2\bar{a} + 2\bar{b}$.
3. If we think of the squares as forming steps where each step has a rise and a run then the run m is such that $m \in \{\lceil b/a \rceil, \lfloor b/a \rfloor\}$. The rise will come from the set $\{\lceil a/b \rceil, \lfloor a/b \rfloor\}$. We disregard floors of size zero.
4. Suppose that $b \geq a$ and $\mathcal{M}$ comes from a decomposition of a general triangle with highlighted nodes as in Figure 5 and b' and a' are the number of highlighted nodes on the base and side of $\mathcal{M}$. In this case b' satisfies $b' \geq \lceil b \rceil - \lceil b/a \rceil$. Further, $b'+2 \geq a'$

Lemma 3.1. *Let $\mathcal{M}$ be a right-triangle mesh as described above where A' and B' are the highlighted nodes of the side and base of $\mathcal{M}$ of size b' and a', respectively.*

- *There exists a path embedding of all nodes in A' to*

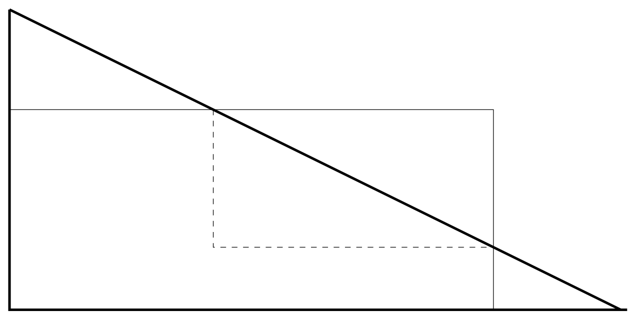

Figure 6: Routings in Right Triangles

all nodes in B' of congestion $O(a'+b')$ and dilation $O(dia)$, where dia is the diameter of $\mathcal{M}$.

- *There exists a path embedding of all nodes in $\mathcal{M}$ to all nodes in B' of congestion $O(dia \cdot b')$ and dilation $O(a+b)$.*

Proof: The proof uses standard embedding techniques. We first show the Lemma holds for the special case when A' and B' are all the nodes on the side and base respectively of $\mathcal{M}$. Here we use what is known as the reflection method. That is, we complete $\mathcal{M}$ to a rectangle of size $\lceil a \rceil$ by $\lceil b \rceil$ and route each pair using exactly one right turn. To route in $\mathcal{M}$, reflect the path about the diagonal of $\mathcal{M}$, see Figure 6. We use the reflection routing for routing both A' to B' and $\mathcal{M}$ to B'. In the case of routing A' to B' it is not too hard to see that the congestion is $O(a'+b')$. In the case of routing $\mathcal{M}$ to B' we get the congestion $O(dia \cdot b')$.

We reduce the general case to the special case. The nodes on the base of $\mathcal{M}$ which are not highlighted sit below at most two steps of $\mathcal{M}$. Let A'' be all the nodes above the leftmost highlighted node of B' (these are shown as open circles in Figure 7).

In the case of routing A' to B' we use the fact that the size of A' and A'' are equal. We route the ith node of A' to the ith node of A'' in a "river" route fashion. We will actually need b' copies of each of these paths. This gives us a congestion of b' in the region between A' and A''. In the region to the right of A'', we use the reflection routing technique as described above for the special case. The congestion bound carries over directly.

In the case of routing $\mathcal{M}$ to B' we first map each element in the region between A' and A'' directly across to an element in A''. We then map the paths to B' using the map defined above for mapping A'' to B'. We will need $b-b'$ copies of each of these paths. Let dia_R be the diameter of the mesh to the right of A''; thus our congestion will be up to constants

$$dia_R \cdot b' + (b-b')(a'+b') \tag{3.2}$$

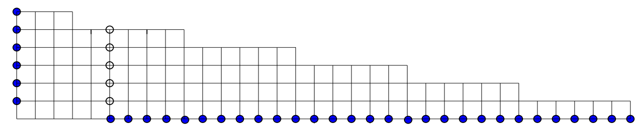

Figure 7: Routing in Right Triangle Mesh

If $a \geq b$ then $b - b' \leq 1$ and thus the result follows. Thus we may assume that $b > a$ we know that this implies that $b' + 2 \geq a'$ Thus $(b-b')(a'+b') \leq (b-b')(2b'+2) \leq 3(b-b')b'$ Therefore Equation 3.2 is bounded by $3(dia \cdot b')$. □

We are now ready to do the inductive proof of Theorem 3.1. Let T be a triangle. As stated earlier we distinguish between the cases when the opposite node of T lies above the base and when it is to one side. If the apex is above the base, the triangle decomposes naturally into two "back to back" right triangles, and we can simply route using the the embeddings established in Lemma 3.1.

Inductively, suppose that T decomposes into some number of right-triangle meshes. Lemma 3.1 establishes the initial case when there is exactly one such triangle. In Figure 8 we have drawn a simple picture for reference. Let $\mathcal{M}_2$ be the rightmost right-triangle mesh of T and $\mathcal{M}_1$ the remaining mesh. There are $|A_2| = a_2$ nodes common to $\mathcal{M}_1$ and $\mathcal{M}_2$, and $|B_2| = b_2$ nodes that may be common with some larger triangle.

We first route in $\mathcal{M}_1$. By the induction hypothesis we can route every node in $\mathcal{M}_1$ to every node in A_2 with congestion bounded by $C(dia_1 \cdot a_2)$ for some constant C. Adjusting the weight of each path by a factor of b_2/a_2 gives the capacity needed to extend them to B_2. Thus the congestion in $\mathcal{M}_1$ is at most $C(dia_1 \cdot b_2) \leq C(dia \cdot b_2)$. It is important to observe that the constant C for these paths, which have already been constructed, doesn't change in this step. The actual size of C is determined by the bounds needed in the next step where we construct new paths in $\mathcal{M}_2$.

We can route all nodes in $\mathcal{M}_2$ to B_2 with congestion $O(dia_2 \cdot b_2)$. Thus we need only show how to route the paths from $\mathcal{M}_1$ on to B_2. For each node in A_2 we have a sum of $|\mathcal{M}_1|\frac{b_2}{a_2}$ paths which need to be routed to B_2. By routing all pairs between A_2 and B_2 we get b_2 paths per node in A_2 from A_2 to B_2. The congestion of the routing is $O(a_2 + b_2)$ by Lemma 3.1. Thus, up to a constant factor, we get a congestion of

$$|\mathcal{M}_1|\frac{b_2}{a_2}\frac{(a_2+b_2)}{b_2} \tag{3.3}$$

We must show that Equation 3.3 is bounded by some constant of $dia \cdot b_2$. We consider two cases depending

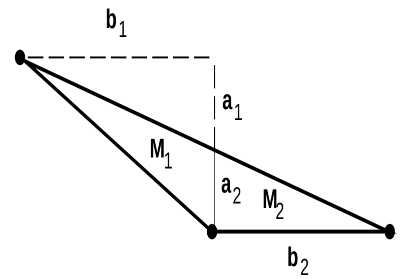

Figure 8: General Triangle for the Proof by Induction

on whether $a_2 \leq b_2$ or not. Suppose that $a_2 \leq b_2$.

$$
\begin{aligned}
|\mathcal{M}_1|\frac{b_2}{a_2}\frac{(a_2+b_2)}{b_2} &\leq 2|\mathcal{M}_1|\frac{b_2}{a_2} \\
&\leq 2(dia_1 \cdot (a_2+2))\frac{b_2}{a_2} \leq 4(dia_1)b_2
\end{aligned}
$$

Here we used the fact that the rise in each step is at most one. Suppose that $a_2 > b_2$. In this case we will use the fact that the run of each step is one.

$$
|\mathcal{M}_1|\frac{b_2}{a_2}\frac{(a_2+b_2)}{b_2} \leq 2|\mathcal{M}_1|
$$

But in this case $dia \cdot (b_2 + 2)$ upper bounds $|\mathcal{M}|$. This completes the proof of Theorem 3.1.

4 Rayleigh Quotient for Complete Bipartite Graphs

The interpolation estimates discussed in Section 5 all involve the minimum of a Raleigh quotient over a subspace of functions satisfying specific boundary conditions. Indeed, in the absence of boundary conditions the Raleigh quotient would be zero. When estimating the Raleigh quotient using graph embeddings, certain vertices on the boundary have a distinguished role in the sense that they encode the boundary data. After considerable trial and error, we found that bipartite graphs provide a very convenient mechanism for capturing the subtleties that distinguish the boundary vertices encoding boundary constraints from other vertices not directly involved with the boundary conditions.

The following lemma characterizes the extreme values of the Rayleigh quotient for the Laplacian of a complete bipartite graph. Let $K_{m,n} = (X, Y, E)$ be the be the complete bipartite graph on the vertex sets X and Y with $|X| = m$ and $|Y| = n$.

LEMMA 4.1. *Let A be the Laplacian of the complete bipartite graph $K_{m,n} = (X, Y, E)$. If the vector $\mathbf{x} \neq \mathbf{0}$ satisfies the constraint $\sum_{i\in X} x_i = 0$ then the the associated Rayleigh quotient satisfies*

$$
\min(m,n) \leq \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \max(m,n)
$$

The extreme values are attained when the components of $\mathbf{x}$ are 0 on either X or Y.

Proof: Letting $\mathbf{x} = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ and A be the Laplacian matrix of the bipartite graph $K_{m,n}$, we compute

$$
\begin{aligned}
\mathbf{x}^T A\mathbf{x} &= \sum_{i\in X, j\in Y} (x_i - y_j)^2 \\
&= \sum_{i\in X, j\in Y} x_i^2 - 2x_iy_j + y_j^2 \\
&= \sum_{i\in X, j\in Y} x_i^2 + \sum_{i\in X, j\in Y} y_j^2 \\
&= n\sum_{i\in X} x_i^2 + m\sum_{j\in Y} y_j^2
\end{aligned}
$$

where the condition $\sum_{i\in X} x_i = 0$ is used to eliminate the cross term. Since $\mathbf{x}^T\mathbf{x} = \sum_{i\in X} x_i^2 + \sum_{j\in Y} y_j^2$ the bounds on the Raleigh quotient follow immediately. It also follows that the Raleigh quotient takes on the value n when all the x_i vanish and the value m when all the y_i vanish. □

We can now show a lower bound for any Raleigh quotient of the Laplacian of a graph which is the 1-skeleton G of a convex rectilinear mesh if we require that the sum of a subset B of the boundary values is zero. Let d be the diameter of G and b the size of B. Further, let $\alpha = c \cdot d^2 \cdot b$ for some constant c as proved in Theorem 3.1 so that there exists a path embedding of $K_{n,b}$ into G of congestion times dilation at most α. It follows that $\alpha A - B$ is positive semi-definite where A and B are the Laplacians of G and $K_{n,b}$, respectively [GLM97].

LEMMA 4.2. *The Raleigh quotient $\frac{x^TAx}{x^Tx} \geq b/\alpha$ when $\sum_{i\in B} x_i = 0$.*

Proof: Let x be any vector such that $\sum_{i\in B} x_i = 0$. Since $\alpha A - B$ is semi-definite we have that $x^T(\alpha A - B)x \geq 0$ and thus

$$
\alpha\frac{x^TAx}{x^Tx} \geq \frac{x^TBx}{x^Tx} \geq b
$$

□

5 Classical Interpolation Theory

Given a simplex T and a function $u : T \rightarrow \Re$, interpolation theory estimates the size of the difference between the function u and a polynomial approximation, u_h, of

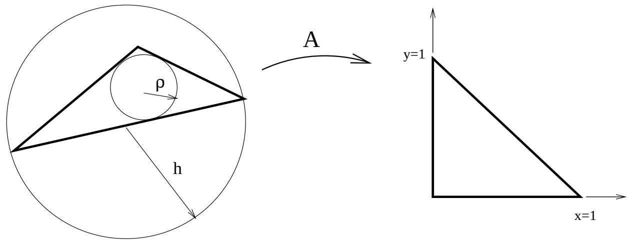

Figure 9: Mapping a Triangle to a Reference Triangle

u defined over T. The classical situation of approximating u on T by the linear function which interpolates u at each of the vertices of T is a basic result found in most finite element texts. In this situation the error, defined by $e = u - u_h$, vanishes at the vertices, and the following lemma is applicable.

LEMMA 5.1. *Let T be a simplex and let $e \in H^2(T)$ vanish at the vertices of T, then there exists a constant $C(T) > 0$ such that*

$$\int_T |De|^2 \leq C(T)^2 \int_T |D^2 e|^2$$

Application of this lemma to $e = u - u_h$ with u_h linear so that $D^2 u_h = 0$ establishes the fundamental estimate

$$\int_T |D(u - u_h)|^2 \leq C(T)^2 \int_T |D^2 u|^2$$

The lemma is typically established using compactness results and gives no estimates of the magnitude of $C(T)$. Scaling arguments can be used to relate the constants of similar simplices. For example, if $h = diam(T)$ is the diameter of T and $\hat{T} = (1/h)T$ is the simplex similar to T having unit diameter, then $C(T) = hC(\hat{T})$.

The classical technique found in finite element texts to bound the constant $C(T)$ is to map any simplex onto a fixed simplex using a linear map $x \mapsto A^{-1}x$ where A is the matrix having columns $x^i - x^0$, where T is the convex hull of the points $\{x^0, x^1, \ldots, x^d\}$, see Figure 9. In this situation it is possible to establish the bound $C(T) \leq C(\hat{T})h(T)^2/\rho(T)$, where $h(T)$ is the diameter of T, $\rho(T)$ is the diameter of the largest inscribed sphere in T, and $\hat{T}$ is the "unit" simplex formed as the convex hull of the origin and the standard unit vectors (see for example Ciarlet [Cia78]).

When all simplices in a finite element mesh have a bounded aspect ratio $h(T)/\rho(T) < C$, this provides uniform estimates for the interpolation error. However, this technique is not optimal. Babuška and Aziz [BA76] showed that in two dimensions the the constant $C(T)$ for any right angle triangle is independent of the aspect ratio; in particular, $C(T) \leq C(\hat{T})h(T)$ where $C(\hat{T})$ is the constant for the "unit" right triangle. By considering linear transformations of right triangles, Babuška and Aziz [BA76] showed that the constant for any triangle of unit diameter depended only upon the maximum angle of a triangle. In particular, if the maximum angle of any triangle in a mesh was bounded away from 180^o then uniform bounds are obtained. However, the actual bound still involves the constant $C(\hat{T})$ associated with the unit right triangle.

While the classical finite element approach is satisfactory for simplicial meshes, problems arise with meshes that may contain arbitrary polytopes. Classical examples of such meshes are the duals of simplicial meshes, which are used for control volume approximations, [MTT$^+$96, Mac53, Nic92]. To apply the finite element technique it would be necessary to map each polytope onto a unit polytope, which, due to the multitude of the number of geometric possibilities, is not feasible. The techniques developed here provide a method of estimating interpolation estimates using tools developed by graph theorists. The constant $C(T)$ appearing in Lemma 5.1 can be characterized as an eigenvalue. Specifically, if $\lambda(T)$ is the minimum Raleigh quotient

$$I(e) = \frac{\int_T |D^2 e|^2}{\int_T |De|^2} \qquad e \in \{u \in H^2(T) \mid u|_{\text{verticies}} = 0\}$$

then $C(T) = 1/\lambda(T)$. In Section 4 we approximated this Raleigh quotient by one associated with the Laplacian of a graph, and use graph embedding techniques to establish lower bounds on λ. Application of these techniques to a triangle enable us to establish the following result which appears to be new.

THEOREM 5.1. *Let $\mathbf{u}$ be a unit vector parallel to the side of a triangle, and let $u \in H^2(T)$ and u_h be the linear function on T that interpolates u at the vertices. Then there a constant C independent of T such that*

$$\int_T |D_{\mathbf{u}}(u - u_h)|^2 \leq C^2 h(T)^2 \int_T |D^2 u|^2$$

where $D_{\mathbf{u}} e = \mathbf{u} \cdot De$ is the directional derivative in the direction $\mathbf{u}$, and $h(T)$ is the diameter of T.

To estimate the derivatives in any direction (and recover the results of Babuška and Aziz) it suffices to write an arbitrary unit vector $\mathbf{u}$ as

$$\mathbf{u} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 \tag{5.4}$$

where $\{\mathbf{u}_i\}_{i=1}^3$ are the unit vectors parallel to the sides of the triangle. In this situation an application of the Cauchy Schwartz inequality yields

$$\int_T |D_{\mathbf{u}}(u - u_h)|^2 \leq 3C^2 |\alpha|^2 h(T)^2 \int_T |D^2 u|$$

The selection of a $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ having minimal length subject to the constraint (5.4) is a classical problem from linear algebra. The minimal length is the reciprocal of the smallest singular value of the matrix $A = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ [GL90], and can be explicitly bounded by $\sqrt{2}/\sin(\theta_{max})$, where θ_{max} is the maximum angle in the triangle. We summarize these estimates in the following corollary.

COROLLARY 5.1. *Let T be a triangle, $u \in H^2(T)$, and u_h be the linear function interpolating u at the vertices of T. Then there exists a constant C independent of T such that*

$$\int_T |D(u - u_h)|^2 \leq \left(\frac{Ch(T)}{\sin(\theta_{max})}\right)^2 \int_T |D^2 u|^2$$

where $h(T)$ is the diameter of T and θ_{max} is the maximum angle.

While this result is not new, we emphasize that our techniques provides explicit estimates of the constant, and are applicable to other polyhedra where classical techniques may fail.

6 Conclusions and Open Questions

Many practitioners believe that finite elements constructed from quadrilateral (in 2 dimensions) or brick (in 3 dimensions) meshes give more accurate solutions to certain elasticity problems than elements constructed from simplices. It is open if the ideas presented here can be extended to quadrilaterals. We can show that using bilinear polynomial interpolates on a rectangle will give good interpolation error[MW98]. Our method uses Fourier analysis and does not seem to generalize to quadrilaterals.

Recently the authors have shown that it is possible to construct good interpolants on "well proportioned" polygons in two dimensions; however, the proof uses functional analytic methods, and extensions of the techniques discussed here would provide significant insight. The extension of this work to tetrahedra in 3D is also open.

The complete and clean characterization of those elements which give good interpolation error is crucial in determining future directions for mesh generation as a whole.

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