GROUPS ACTING ON REGULAR GRAPHS AND GROUP AMALGAMS

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0. INTRODUCTION

The s-transitive groups acting on connected regular graphs have been studied by several people: W.T. Tutte, A. Gardiner, N. Biggs, R.M. Weiss, the authors and others. The purpose of this paper is two-fold. First we establish that the problems about constructing and existence of such groups have a simple purely group-theoretical formulation in terms of group amalgams. Second, we determine the group amalgams associated with the locally regular groups operating on cubic graphs.

In Section 1 we introduce graph-theoretical concepts that we need and in Section 2 the basic definitions about group amalgams. Section 3 explains the connection between 1-transitive groups and group amalgams. Section 4 classifies finite simple amalgams of degree (3,2). It is surprising (Theorem 4) that there are precisely 7 such amalgams. Section 5 expresses some known results about s-transitive groups in the language of group amalgams. Section 7 explains the connection between locally 1-transitive groups and group amalgams. The next two sections deal with the amalgams associated to locally regular groups acting on cubic graphs. We show that there is essentially only one such amalgam for locally s-regular groups $s = 1,2,3,4,5$, or 7.

We end with two remarks, in the last section, which illustrate the usefullness of the graph-theoretical method in the study of group amalgams.

1. GROUPS ACTING ON GRAPHS

Let G be a graph having neither loops nor multiple edges. Let A be a group and $f: A \to \mathrm{Aut}(G)$ a homomorphism. Then we say that A _acts_ on G and we write $f(a)(x) = a \cdot x$ for $a \in A$ and x a vertex of G. If f is injective we say that the action is _faithful_ and in that case we may consider A as a subgroup of Aut(G).

From now on we consider a pair (G,A) where G is _a connected graph and_ A _a subgroup of_ Aut(G). Recall that an s-arc S of G is a map

$$S:\{0,1,\ldots,s\} \to V(G) = V$$

(V(G) = the vertex-set of G) such that S(i) and S(i + 1) are adjacent for $0 \le i \le s - 1$ and $S(i) \ne S(i + 2)$ for $0 \le i \le s - 2$.

We say that A is s-_transitive_ (resp. _locally_ s-_transitive_) if given any two s-arcs S_1 and S_2 (resp. two s-arcs S_1 and S_2 satisfying $S_1(0) = S_2(0)$), there exists $\alpha \in A$ such that $\alpha \circ S_1 = S_2$. If A is locally s-transitive ($s \ge 1$) then A acts transitively on the set of edges of G [1, Lemma 1]. Thus, in this case, A is either transitive on V or it has precisely two orbits, say V^+ and V^-, and G is bipartite with partition (V^+,V^-). In the first case A is in fact s-transitive.

We say that A is s-<u>regular</u> (resp. <u>locally</u> s-<u>regular</u>) if A is s-transitive (resp. locally s-transitive) and for each s-arc S and $\alpha \in A$ the equality $\alpha \circ S = S$ implies $\alpha = 1$.

We shall say that G is s-<u>transitive</u>, <u>locally</u> s-<u>transitive</u>, etc. if Aut(G) has the corresponding property. If G is s-transitive $(s \geq 0)$ then it is a regular graph, i.e., every vertex of G has the same degree (or valence). If G is locally s-transitive $(s \geq 1)$ then it is either regular or bipartite with partition (V^+, V^-) of V and any two vertices in V^+ (resp. V^-) have the same degree. A regular graph of degree 3 is also called <u>cubic graph</u>.

We say that A has <u>index</u> s (resp. <u>local index</u> s) if A is s-transitive (resp. locally s-transitive) but it is not (s+1)-transitive (resp. locally (s+1)-transitive). It is well-known [5] that if G is a cubic graph and A has index s $(1 \leq s < \infty)$ then A is s-regular and s takes one of the values 1,2,3,4,5. We remark that the corresponding statement is not true for locally s-transitive groups. For instance if $G = K_{3,3}$ (the complete bipartite graph on 6 vertices with three vertices in each part) then Aut(G) has a subgroup isomorphic to $C_3 \times S_3$ which has local index 1 and is not locally 1-regular.

We recall that R.M. Weiss [7] has shown that if G is a cubic graph and A is locally s-regular $(1 \leq s < \infty)$ then s takes one of the values 1,2,3,4,5,7.

2. GROUP AMALGAMS

A <u>group amalgam</u> is a triple $(X,Y;H)$ where X,Y,H are groups, H is a subgroup of both X and Y and $X \cap Y = H$. Since X and Y determine H we shall write sometimes (X,Y) instead of $(X,Y;H)$.

A <u>morphism</u> from an amalgam $(X_1,Y_1;H_1)$ to the amalgam $(X_2,Y_2;H_2)$ is a pair (f_1,f_2) of group homomorphisms

$$f_1:X_1 \to X_2, \quad f_2:Y_1 \to Y_2$$

such that f_1 and f_2 coincide on H_1 and

$$f_1^{-1}(H_2) = f_2^{-1}(H_2) = H_1.$$

We say that an amalgam $(X',Y';H')$ is a <u>subamalgam</u> of the amalgam $(X,Y;H)$ if X' (resp. Y') is a subgroup of X (resp. Y) and $X' \cap H = Y' \cap H = H'$. We say that this subamalgam is <u>transitive</u> if $X'H = X$ and $Y'H = Y$.

Let $(X,Y;H)$ be an amalgam and $a \in H$. Let $f_1(x) = axa^{-1}$ $(x \in X)$ and $f_2(y) = aya^{-1}$ $(y \in Y)$. Then (f_1,f_2) is an automorphism of $(X,Y;H)$ which is called <u>inner automorphism</u>. Two subamalgams (X',Y') and (X'',Y'') of (X,Y) are said to be <u>conjugate</u> if there exists an inner automorphism (f_1,f_2) of (X,Y) such that $f_1(X') = X''$ and $f_2(Y') = Y''$.

A <u>normal subgroup</u> of $(X,Y;H)$ is, by definition, a subgroup N of H which is normal in both X and Y. By Zorn's lemma, there exists a unique maximal normal subgroup of $(X,Y;H)$ which is called <u>the core</u> of this amalgam. If the core is trivial (i.e., the identity subgroup) then we say that the amalgam is <u>simple</u>.

If K is a normal subgroup of $(X,Y;H)$ then $(X/K,Y/K;H/K)$ is also an amalgam called the quotient of $(X,Y;H)$ modulo K. If K is the core of $(X,Y;H)$ then this quotient is a simple amalgam.

If (X_i,Y_i), $i = 1,2$ are amalgams then a morphism $(f_1,f_2):(X_1,Y_1) \to (X_2,Y_2)$ is called an imbedding if both f_1 and f_2 are injective. Two imbeddings (f_1,f_2) and (g_1,g_2) of (X_1,Y_1) into (X_2,Y_2) are called equivalent if the subamalgams $(f_1(X_1),f_2(Y_1))$ and $(g_1(X_1),g_2(Y_1))$ are conjugate in (X_2,Y_2).

An amalgam (X,Y) is finite if both X and Y are finite groups. The degree of an amalgam $(X,Y;H)$ is the ordered pair (m,n) where m (resp. n) is the index of H in X (resp. Y).

If $(X,Y) = (X,Y;H)$ is an amalgam then we denote by

$$X \underset{H}{*} Y$$

the generalized free product of X and Y amalgamating the subgroup H.

An amalgam is proper if its degree (m,n) is such that $m \geq 2$ and $n \geq 2$.

We remark that an amalgam $(X,Y;H)$ may be simple without the group $X \underset{H}{*} Y$ being simple. This is always the case if the amalgam is proper and finite.

3. AMALGAMS OF 1-TRANSITIVE GROUPS

We consider pairs (G,A) where G is a connected regular graph of valence d, and A a subgroup of $\mathrm{Aut}(G)$ which is 1-transitive.

Fix a 1-arc S in G, $S(0) = a$, $S(1) = b$; thus $\{a,b\}$ is an edge of G.

Define $X = A(a)$ = the fixer in A of the vertex a. Let Y be the subgroup of A consisting of all $\alpha \in A$ such that $\{\alpha(a),\alpha(b)\} = \{a,b\}$. Then $H = X \cap Y$ is the fixer in A of the 1-arc S.

Hence we can associate to (G,A) and S the group amalgam $(X,Y;H)$. Since A is 1-transitive this group amalgam is in fact independent (up to isomorphism) of the choice of S. Indeed, if T is another 1-arc, $T(0) = u$, $T(1) = v$ and $\alpha \in A$ is such that $\alpha \circ S = T$ then $\alpha X\alpha^{-1} = A(a)$ and $\alpha Y\alpha^{-1}$ is the stabilizer in A of the edge $\{u,v\}$.

<u>Theorem 1</u>. Let (G,A), S, and $(X,Y;H)$ be as above. Then A is generated by X and Y, the amalgam $(X,Y;H)$ is simple and has degree $(d,2)$.

<u>Proof</u>: The first assertion follows from [3, Proposition 1]. Let N be a normal subgroup of $(X,Y;H)$. Thus $N \subset H$ and since X and Y generate A, we have $N \triangleleft A$. Now let $x \in V = V(G)$ and choose $\alpha \in A$ such that $x = \alpha(a)$. If $\beta \in N$ then

$$\beta(x) = \beta\alpha(a) = \alpha(\alpha^{-1}\beta\alpha(a)) = \alpha(a) = x$$

because $\alpha^{-1}\beta\alpha \in N \subset H \subset A(a)$. Hence $\beta(x) = x$ for all $x \in V$ and so $\beta = 1$ and $N = \{1\}$. Thus $(X,Y;H)$ is a simple amalgam.

$X = A(a)$ acts transitvely on the set of d neighbours of a and H is the fixer in X of the vertex b. Therefore $(X:H) = d$. Similarly, $(Y:H) = 2$.

Let (G_i, A_i), $i = 1,2$ be two pairs as (G,A) above. We say that they are of the <u>same type</u> if the associated amalgams $(X_i, Y_i; H_i)$, $i = 1,2$ are isomorphic.

The converse of Theorem 1 is also valid.

<u>Theorem 2</u>. Let $(X,Y;H)$ be a simple amalgam of degree $(d,2)$. Then there exists a pair (G,A) consisting of a connected regular graph G of valence d and a subgroup A of $\mathrm{Aut}(G)$ which is 1-transitive such that the amalgam associated to (G,A) is isomorphic to $(X,Y;H)$.

<u>Proof</u>: Let A be any group containing the amalgam $(X,Y;H)$ which is generated by X and Y and such that

$$y^{-1}Xy \cap X = H \qquad (y \in Y,\ y \notin H).$$

For instance, we can take $A = X \underset{H}{*} Y$.

Define G as follows: its vertices are the cosets uX, $u \in A$ and its edges are $\{uX, uyX\}$ for $u \in A$, $y \in Y$, $y \notin H$. Since X and Y generate A, the graph G is connected. The group A acts on G by left translations and it is transitive on $V = V(G) = A/X$. Therefore, G is a regular graph. The neighbours of the vertex X in G are the vertices xyX where $x \in X$ and $y \in Y \setminus H$ is fixed. The number of these vertices is equal to the index of $y^{-1}Xy \cap X = H$ in $y^{-1}Xy$, i.e., it is equal to

$$(y^{-1}Xy : H) = (y^{-1}Xy : y^{-1}Hy) = (X:H) = d.$$

Let S be the 1-arc of G defined by $S(0) = X$, $S(1) = yX$. The subgroup X of A is the fixer of the vertex $X \in V$. Similarly, the fixer in A of the vertex yX is the subgroup yXy^{-1} of A. The fixer of the 1-arc S is the subgroup

$$X \cap yXy^{-1} = yHy^{-1} = H.$$

Hence the kernel of the homomorphism $A \to \mathrm{Aut}(G)$ is contained in H. Since $(X,Y;H)$ is a simple amalgam we conclude that the action is faithful and hence we may consider A as a subgroup of $\mathrm{Aut}(G)$.

Since X permutes transitively the neighbours xyX $(x \in X)$ of $X \in V$ in G, it follows that A is 1-transitive. Thus we have a pair (G,A) where G is a connected regular graph of valence d and $A \leq \mathrm{Aut}(G)$ is 1-transitive. The amalgam associated to (G,A) and the 1-arc S is precisely the original amalgam $(X,Y;H)$.

Let $(X,Y;H)$ be a simple amalgam of degree $(d,2)$, $d > 3$. Choose $x \in X \setminus H$, $y \in Y \setminus H$ and put $z = xy$. Now define $H_1 = H$,

$$H_{i+1} = H_i \cap zH_iz^{-1} \quad (i \geq 1).$$

<u>Theorem 3</u>. Let (G,A) be as before with associated amalgam $(X,Y;H)$. Then the index of A is the smallest integer s (or ∞ otherwise) such that $(H_s : H_{s+1}) < d - 1$.

<u>Proof</u>: Let S be the 1-arc used in the construction of $(X,Y;H)$ and $S(0) = a$, $S(1) = b$. From our choice of x and y above, we have $x(a) = a$, $x(b) = c \neq b$, $y(a) = b$, $y(b) = c$. Thus $z(b) = xy(b) = x(a) = a$, $z(a) = xy(a) = x(b) = c$, i.e., in the terminology of [3], z is an A-shunting.

Let us define $a_0 = b$, $a_1 = a$ and in general $a_{i+1} = z(a_i)$ $(i \geq 0)$. Note that $H = H_1 = A(a_0, a_1)$ is the fixer of the 1-arc (a_0, a_1). Hence zHz^{-1} is the fixer of the 1-arc (a_1, a_2) and consequently $H_2 = H_1 \cap zH_1z^{-1}$ is the fixer in A of the 2-arc (a_0, a_1, a_2). In general, H_i is the fixer of the i-arc $(a_0, a_1, \ldots, a_i)$.

Let s be the index of A. For $i < s$ the equality $(H_i : H_{i+1}) = d-1$ is valid since H_i acts transitively on the neighbours of a_i distinct from a_{i-1} and H_{i+1} is the fixer in H_i of a_{i+1}. Since A is not $(s+1)$ - transitive, the same argument shows that $(H_s : H_{s+1}) < d-1$.

In view of this theorem we define the <u>index</u> of a simple amalgam $(X,Y;H)$ of degree $(d,2)$, $d \geq 3$ to be the smallest integer s such that $(H_s : H_{s+1}) < d-1$. The subgroups H_i are defined as above and it follows from Theorem 3 that this s is independent of the choice of x and y.

If $(X,Y;H)$ is a simple amalgam of degree $(d,2)$, $d \geq 3$ and index s and if $H_s = \{1\}$, or equivalently the order of H is $(d-1)^{s-1}$, then we shall say that this amalgam is s-<u>regular</u> or just <u>regular</u>.

4. FINITE SIMPLE AMALGAMS OF DEGREE (3,2)

We shall now define seven important amalgams, see [3, Section 7]. We denote by C_n (resp. D_n) the cyclic (resp. dihedral) group of order n (resp. $2n$). The amalgams will be defined by generators and relations; it is to be understood that for each X,Y,H one should use only the relations which involve only the generators of that group.

We now list our amalgams:

$Am(1') = (X_1, Y_1; H_1)$, $X_1 = \langle a \rangle$, $Y_1 = \langle y \rangle$, $a^3 = y^2 = 1$, $H_1 = \{1\}$.

$Am(2') = (X_2, Y_2'; H_2)$, $X_2 = \langle a,b \rangle$, $Y_2' = \langle b,y \rangle$, $H_2 = \langle b \rangle$,

$$a^2 = b^2 = y^2 = (yb)^2 = (ab)^3 = 1.$$

Thus $X_2 \cong D_3$, $Y_2' \cong C_2 \times C_2$, $H_2 \cong C_2$.

$Am(2'') = (X_2, Y_2''; H_2)$, $X_2 = \langle a,b \rangle$, $Y_2'' = \langle y \rangle$, $H_2 = \langle b \rangle$, $a^2 = b^2 = y^4 = 1$,

$$y^2 = b, \quad (ab)^3 = 1.$$

Thus $Y_2'' \cong C_4$.

$Am(3') = (X_3, Y_3; H_3)$, $X_3 = \langle a,b,c \rangle$, $Y_3 = \langle b,c,y \rangle$, $H_3 = \langle b,c \rangle$,

$$y^2 = a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ac)^3 = 1, \quad yby = c.$$

Thus $X_3 \cong D_6$, $Y_3 \cong D_4$, $H_3 \cong C_2 \times C_2$.

$Am(4') = (X_4, Y_4'; H_4)$, $X_4 = \langle a,b,c,d \rangle$, $Y_4' = \langle b,c,d,y \rangle$, $H_4 = \langle b,c,d \rangle$,

$$y^2 = a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = 1,$$
$$(ac)^2 = b, \quad (bd)^2 = c, \quad (ad)^3 = 1, \quad (yc)^2 = 1, \quad yby = d.$$

Thus $X_4 \cong S_4$ (symmetric group), $Y_4' \cong D_8$, and $H_4 \cong D_4$.

$Am(4'') = (X_4, Y_4''; H_4)$, $X_4 = \langle a,b,c,d \rangle$, $Y_4'' = \langle b,c,d,y \rangle$, $H_4 = \langle b,c,d \rangle$,

$$a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = 1,$$
$$(ac)^2 = b, \quad (bd)^2 = c, \quad (ad)^3 = 1, \quad y^2 = c, \quad yby^{-1} = d.$$

Thus Y_4'' is the so called quasi-dihedral group of order 16.

$Am(5') = (X_5, Y_5; H_5)$, $X_5 = \langle a,b,c,d,e \rangle$, $Y_5 = \langle b,c,d,e,y \rangle$, $H_5 = \langle b,c,d,e \rangle$,

$$y^2 = a^2 = b^2 = c^2 = d^2 = e^2 = (ab)^2 = (bc)^2 = (cd)^2$$
$$= (de)^2 = (ac)^2 = (bd)^2 = (ce)^2 = 1,$$
$$(ad)^2 = bc, \quad (be)^2 = cd, \quad (ae)^3 = 1, \quad yby = e, \quad ycy = d.$$

Thus $X_5 = \langle a,bc,cd,e \rangle \times \langle c \rangle$, $\langle a,bc,cd,e \rangle \cong S_4$, $\langle c \rangle \cong C_2$,

$Y_5 = \langle b,c,d,e \rangle \rtimes \langle y \rangle$ (semidirect product), and

$H_5 = \langle b,e \rangle \times \langle c \rangle = \langle b,e \rangle \rtimes \langle d \rangle$, $\langle b,e \rangle \cong D_4$.

Theorem 4. Every finite simple amalgam of degree (3,2) is isomorphic to one of the seven amalgams listed above.

Proof: Let (X,Y;H) be such an amalgam. By Theorem 2 there exists a connected cubic graph G and $A \le Aut(G)$ which is 1-transitive such that (X,Y;H) is the amalgam associated to (G,A). Then A must be s-regular for some s = 1,2,3,4,5 and our assertion follows from [3, Proposition 15].

Theorem 5. The number of subamalgams of Am(s') or Am(s") which are isomorphic to Am(t') or Am(t") is given in the table below.

t \ s	1'	2'	2"	3'	4'	4"	5'
1'	1	2	0	2	16	0	16
2'		1	0	1	0	0	0
2"		0	1	1	0	0	0
3'				1	0	0	0
4'					1	0	1
4"					0	1	1
5'							1

Proof: This follows from [3, Theorems 3 and 4].

We have shown in [3] that the two subamalgams Am(1') are not conjugate in Am(2') but they are conjugate in Am(3'). The 16 subamalgams Am(1') in Am(4') split into 2 conjugacy classes, each of size 8. Inside Am(5') all these 16 subamalgams are conjugate.

5. FINITE SIMPLE AMALGAMS OF DEGREE $(d,2)$, $d > 3$.

Very little is known about such amalgams. We shall summarize here the most important results.

First we show that there are infinitely many finite simple amalgams of degree $(4,2)$. Let H be an elementary abelian group of order 2^{n+1} with a basis $a_0, a_1, \ldots, a_n$. Let H' be the subgroup of H generated by $a_0, a_1, \ldots, a_{n-1}$. We take X to be the semidirect product $X = H' \rtimes \langle a_n, x\rangle$ where $\langle a_n, x\rangle \cong D_4$, $x^2 = a_n^2 = 1$, $(xa_n)^4 = 1$. The action of a_n on H' is trivial and x acts as follows: $xa_ix = a_{n-1-i}$ $(0 \le i \le n-1)$. Finally, let Y be the semidirect product $Y = H \rtimes \langle y\rangle$ where $\langle y\rangle \cong C_2$, $y^2 = 1$ and y acts on H as follows $ya_iy = a_{n-i}$ $(0 \le i \le n)$. Then $(X,Y;H)$ is a finite amalgam of degree $(4,2)$ and it is simple. Indeed, let $N \subset H$ be a subgroup which is normal in both X and Y. Then

$$N \subset H \cap xHx = \langle a_0, a_1, \ldots, a_{n-1}\rangle = H'$$

$$N \subset H' \cap yH'y = \langle a_1, a_2, \ldots, a_{n-1}\rangle.$$

By repeating this argument we obtain that $N = \{1\}$.

The next Theorem we deduce from a result of R.M. Weiss [8].

Theorem 6. Let $(X,Y;H)$ be a finite simple amalgam of degree $(d,2)$ where d is a prime > 3. If X is solvable then its order divides $d(d-1)^2$ and consequently the index of $(X,Y;H)$ is ≤ 2.

Proof: By Theorem 2 there exists a pair (G,A) where G is a connected regular graph of valence d and $A \leq Aut(G)$ is 1-transitive such that the amalgam associated with (G,A) is isomorphic to $(X,Y;H)$. Now we can proceed as in the proof of R.M. Weiss of his Satz in [8].

In the next theorem we collect some know pastial results about amalgams of degree $(d,2)$, $d > 3$.

Theorem 7. Let $(X,Y;H)$ be a finite simple amalgam of degree $(d,2)$, $d > 3$ and index s. Then

(i) if $d = 4$ we have $s = 1,2,3,4,7$;

(ii) if $d - 1$ is a prime ≥ 5 we have $s = 1,2,3,4$;

(iii) if $d - 1$ is a prime ≥ 3 and the amalgam is regular we have $s = 1,2,3$ and if $s = 2,3$ then $d - 1$ is a Mersenne prime;

(iv) if $d = np^r + 1$ where $n < p$, p a prime, and the amalgam is regular, then $s = 1,2,3,4,5,7$.

The parts (i) and (ii) are due to A. Gardiner [4], (iii) to the first author [2], and (iv) to Weiss [6].

6. AMALGAMS OF LOCALLY 1-TRANSITIVE GROUPS

Let G be a connected bipartite graph, $V = V(G)$ its vertex set and (V^+,V^-) the corresponding partition of V. Let $A \leq Aut(G)$ be locally 1-transitive. In order to avoid duplication with the case of 1-transitive groups <u>we assume here that</u> A <u>preserves both</u> V^+ <u>and</u> V^-. Thus V^+ and V^- are the two orbits of A in V. It follows that every vertex in V^+ has the same degree d^+ and every vertex in V^- has degree d^-.

A has also two orbits in the set of all 1-arcs of G. If S_i, $i = 1,2$ are two 1-arcs then there exists an $\alpha \in A$ such that $\alpha \circ S_1 = S_2$ iff $S_1(0)$ and $S_2(0)$ are both in V^+ or both in V^-.

Fix a 1-arc S such that $S(0) = a \in V^+$ and $S(1) = b \in V^-$. Let $X = A(a)$ be the fixer in A of the vertex a and let $Y = A(b)$. Then $H = X \cap Y = A(a,b)$ is the fixer in A of S. We associate with (G,A) and S the group amalgam $(X,Y;H)$. Since A is transitive on 1-arcs T such that $T(0) \in V^+$, it follows that this amalgam is independent (up to isomorphism) of the choice of S.

<u>Theorem 8</u>. Let (G,A), S and $(X,Y;H)$ be as above. Then A is generated by X and Y, the amalgam $(X,Y;H)$ is simple and has degree (d^+,d^-).

<u>Proof</u>: Let $B = \langle X,Y\rangle$. In order to show that $B = A$ it suffices to prove that B is transitive on V^+ and on V^-. Let $u \in V^+$ and we shall prove that there exists $\alpha \in A$ such that $\alpha(u) = a$. We shall prove this by induction on the distance $\delta(a,u) = d$ from a to u. If $d = 0$ then $u = a$ and we can take $\alpha = 1$. Let $d > 0$ and assume that our

claim is true for vertices in V^+ whose distance from a is less than d. If $d(b,u) = d + 1$ we can choose $\beta \in X$ such that $d(b,\beta(u)) = d - 1$. Of course we still have $d(a,\beta(u)) = d$. This shows that we can assume that $d(b,u) = d - 1$. Then there exists $\gamma \in Y$ such that $d(a,\gamma(u)) = d - 2$ and we can then use the induction hypothesis. Thus A is generated by X and Y.

Since X acts transitively on the d^+ neighbours of a and H is the fixer in A of b, it follows that $(X:H) = d^+$. Similarly, $(Y:H) = d^-$. Finally, the simplicity of $(X,Y;H)$ follows as in the proof of Theorem 1.

The converse is also valid.

Theorem 9. Let $(X,Y;H)$ be a simple amalgam of degree (d^+,d^-). Then there exists a connected bipartite graph G with the underlying partition (V^+,V^-) of $V = V(G)$ and a subgroup $A \leq Aut(G)$ which is 1-transitive and such that the amalgam associated with (G,A) is isomorphic to $(X,Y;H)$.

Proof: Let A be a group containing the amalgam $(X,Y;H)$ which is generated by X and Y. For instance we can take $A = X \underset{H}{*} Y$.

Define G as follows: its vertex-set V is the disjoint union of $V^+ = G/X$ and $V^- = G/Y$ and the edges of G are $\{uX,uY\}$ for $u \in A$. It is easy to check that uX and vY are adjacent iff $u^{-1}v \in XY$. Since X and Y generate A, G is connected and bipartite with associated partition (V^+,V^-). The group A acts on G by left translations and V^+ and V^- are the two orbits of A in V. Hence any two vertices in V^+ (resp. V^-) have the same degree. The neighbours of the vertex X are the vertices xY, $x \in X$. The number of these vertices is equal to the index $(X:X \cap Y) = (X:H) = d^+$. Similarly, the vertex Y has d^- neighbours.

Let S be the 1-arc $S(0) = X$, $S(1) = Y$. The fixer of the vertex X (resp. Y) in A is the subgroup X (resp. Y) of A. Hence the kernel of the homomorphism $A \to \mathrm{Aut}(G)$ is contained in $H = X \cap Y$. Since $(X,Y;H)$ is a simple amalgam we conclude that A acts faithfully on G and hence we may consider A as a subgroup of $\mathrm{Aut}(G)$.

Since the group X (resp. Y) permutes transitively the neighbours of the vertex X (resp. Y), it follows that A is locally 1-transitive. The amalgam associated to (G,A) and the 1-arc S is precisely the original amalgam $(X,Y;H)$.

We continue to use the notation and hypotheses from the beginning of this section. Fix an $x \in X \setminus H$ and $y \in Y \setminus H$. Then we define vertices a_i by

$$a_0 = a, \quad a_1 = b,$$
$$a_{-i} = x(a_i), \quad (i \geq 1),$$
$$a_{i+2} = y(a_{-i}), \quad (i \geq 0).$$

Define
$$H_1^+ = H_1^- = H$$
and
$$H_{i+1}^+ = H_i^+ \cap yH_i^- y^{-1}, \quad i \geq 1;$$
$$H_{i+1}^- = H_i^- \cap xH_i^+ x^{-1}, \quad i \geq 1.$$

Theorem 10. Use the above hypotheses and notation. Then the local index of A is the smallest integer s (or ∞ if no such s exists) such that

$$(H_s^+ : H_{s+1}^+) < \begin{cases} d^+ - 1 & \text{if } s \text{ is even,} \\ d^- - 1 & \text{if } s \text{ is odd,} \end{cases}$$

or

$$(H_s^- : H_{s+1}^-) < \begin{cases} d^- - 1 & \text{if } s \text{ is even,} \\ d^+ - 1 & \text{if } s \text{ is odd.} \end{cases}$$

Proof: Note that H is the fixer in A of the 1-arc $(a_0, a_1) = (a, b)$. Hence, xHx^{-1} is the fixer of the 1-arc (a_{-1}, a_0) and so H_2^- is the fixer of the 2-arc (a_{-1}, a_0, a_1). Similarly, H_2^+ is the fixer of (a_0, a_1, a_2). In general, H_i^- is the fixer of the i-arc $(a_{-i+1}, \ldots, a_0, a_1)$ and H_i^+ is the fixer of the i-arc $(a_0, a_1, \ldots, a_i)$.

Let s be the local index of A. If $i < s$ then H_i^+ acts transitively on the neighbours of a_i distinct form a_{i-1}. If i is even then $a_i \in V^+$ and so $(H_i^+ : H_{i+1}^+) = d^+ - 1$. If i is odd then $a_i \in V^-$ and so $(H_i^+ : H_{i+1}^+) = d^- - 1$.

Similarly, we have

$$(H_i^- : H_{i+1}^-) = \begin{cases} d^- - 1 & \text{if } i \text{ is even,} \\ d^+ - 1 & \text{if } i \text{ is odd.} \end{cases}$$

Since A is not locally $(s+1)$-transitive, the same argument shows that for $i = s$ at least one of the inequalities given in the theorem must hold.

In view of this theorem we define the local index of a simple amalgam $(X,Y;H)$ of degree (d^+, d^-) to be the smallest integer s such that one of the inequalities in Theorem 10 is valid. It follows from our theorem that s is independent of the choice of x and y.

If $(X,Y;H)$ is a simple amalgam of degree (d^+,d^-), $d^+ \geq 2$, $d^- \geq 2$ and local index s and if $H_s^+ = H_s^- = \{1\}$, then we shall say that this amalgam is locally s-regular or just locally regular.

7. FINITE LOCALLY REGULAR SIMPLE AMALGAMS OF DEGREE (3,3).

Let $(X,Y;H)$ be such an amalgam and let s be its local index. By Theorem 9 there exists a bipartite connected cubic graph G with associated partition (V^+,V^-) of $V = V(G)$ and a subgroup $A \leq Aut(G)$ which preserves V^+ and V^- such that the amalgam associated to (G,A) is isomorphic to $(X,Y;H)$. Say, X is the fixer of a vertex $a \in V^+$ and Y is the fixer of a neighbouring vertex $b \in V^-$.

Since the amalgam $(X,Y;H)$ is locally s-regular for some positive integer s, it follows that the group A is locally s-regular and V^+ and V^- are two orbits of A in V. It was proved by Weiss [7] that we must have $s = 1,2,3,4,5$, or 7.

Theorem 11. For each $s = 1,2,3,4,5$ there exists precisely one locally s-regular finite simple amalgam of degree $(3,3)$. For $s = 7$ there are two such amalgams $(X,Y;H)$ and $(Y,X;H)$ which differ only in the order of X and Y.

The proof will be given separately for each value of s.

Case s = 1. The local 1-regularity implies that $H = \{1\}$ and so $X \cong C_3$ and $Y \cong C_3$.

Case s = 2. Now H has order 2, so $H \cong C_2$. If $x \in X \setminus H$ then $H_2^- = H \cap xHx^{-1}$ satisfies $(H:H_2) = 2$. Hence X is a non-abelian group of order 6 and the same is true for Y. Thus we have $X \cong Y \cong D_3$, $H \cong C_2$.

Case s = 3. Let (G,A) be constructed from our amalgam as in the beginning of this section. Let $\tilde{b}$ be the generator of the fixer $A(a,b,c)$ of the 2-arc (a,b,c) which is cyclic of order 2, see Figure 1.

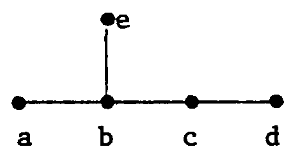


Figure 1

The element $\tilde{b}$ fixes b and its 3 neighbours and moves each vertex at distance 2 from b. It is a unique such element in A and hence $\tilde{b}$ belongs to the center of $A(b) = Y$, say.

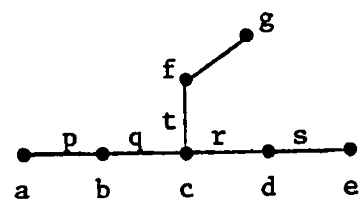
Since $\tilde{a}(c) = e$ we have $\tilde{a}\tilde{c}\tilde{a} = \tilde{e}$. Thus $(\tilde{a}\tilde{c})^2 = \tilde{e}\tilde{c}$ which implies that $\tilde{a}\tilde{c}$ is conjugate to $(\tilde{a}\tilde{c})^2$. Since $(\tilde{a}\tilde{c})^3 \in A(a,b,c)$ we have either $(\tilde{a}\tilde{c})^3 = 1$ or $(\tilde{a}\tilde{c})^3 = \tilde{b}$. The second case is impossible since $\tilde{a}\tilde{c}$ and $(\tilde{a}\tilde{c})^2$ have the same order. Hence $(\tilde{a}\tilde{c})^3 = 1$.

Thus $A(b) = Y = \langle\tilde{a},\tilde{b},\tilde{c}\rangle = \langle\tilde{a},\tilde{c}\rangle \times \langle\tilde{b}\rangle \cong D_3 \times C_2 \cong D_6$.

Similarly $A(a) = X \cong D_6$. Further we have $H = X \cap Y = A(a,b) = \langle\tilde{a},\tilde{b}\rangle \cong C_2 \times C_2$.

Clearly this determines uniquely our amalgam.

<u>Case</u> s = 4. The fixer A(b,c) has order 8, see Figure 2.



<u>Figure 2</u>

Let α be in the center of A(b,c). We claim that α fixes a and d. Indeed, say that $\alpha(d) = f$, $\alpha(e) = g$. Let $\beta \neq 1$ be in A(b,c,d,e). Then since α and β commute it follows that β also fixes $f = \alpha(d)$ and $g = \alpha(e)$. Thus β fixes the 4-arc (e,d,c,f,g) and since A is locally 4-regular one must have $\beta = 1$, a contradiction.

Thus $\alpha \in A(a,b,c,d)$ and so the center of A(b,c) is of order 2 and we shall write $\alpha = \tilde{q}$ where q is the edge {b,c}. The element $\tilde{q}$ is of order 2 and fixes all neighbouring vertices while it moves every vertex at distance 2 from q. Such an element is defined for every edge of G.

Since $\tilde{p}(q) = q$ we have $\tilde{p}\tilde{q}\tilde{p} = \tilde{q}$ i.e., $\tilde{p}$ and $\tilde{q}$ commute. On the other hand $\tilde{p}(r) = t$ and so $\tilde{p}\tilde{r}\tilde{p} = \tilde{t}$.

We have

$$A(b,c,d) = \langle \tilde{q},\tilde{r} \rangle \cong C_2 \times C_2,$$

$$A(a,b,c) = \langle \tilde{p},\tilde{q} \rangle \cong C_2 \times C_2.$$

Also $$A(b,c) = \langle \tilde{p},\tilde{q},\tilde{r} \rangle \cong D_4$$

because $(\tilde{p}\tilde{r})^2 = \tilde{t}\tilde{r} = \tilde{q}$. The equality $\tilde{t}\tilde{r} = \tilde{q}$ follows from the fact that

$A(b,c,d) = A(b,c,f) = A(f,c,d) \cong C_2 \times C_2$ and that $\tilde{q}$, $\tilde{r}$, $\tilde{t}$ are the 3 non-identity elements in this group.

Now we have

$$A(c) = \langle\tilde{p},\tilde{q},\tilde{r},\tilde{s}\rangle.$$

Since $\widetilde{p}\widetilde{s}(d) = \tilde{p}(d) = f$, $\widetilde{p}\widetilde{s}(f) = \tilde{p}(b) = b$, and $\widetilde{p}\widetilde{s}(b) = \tilde{p}(f) = d$ we have $(\widetilde{p}\widetilde{s})^3 \in A(b,c,d)$. Thus the order of $\widetilde{p}\widetilde{s}$ is either 3 or 6. But as in the previous case we have that $\widetilde{p}\widetilde{s}$ and $(\widetilde{p}\widetilde{s})^2$ are conjugate in A, so that we must have $(\widetilde{p}\widetilde{s})^3 = 1$.

Thus

$$A(c) = \langle\tilde{q},\tilde{r}\rangle \rtimes \langle\tilde{p},\tilde{s}\rangle,$$

$\langle\tilde{q},\tilde{r}\rangle \cong C_2 \times C_2$, $\langle\tilde{p},\tilde{s}\rangle \cong D_3$ and $\langle\tilde{p},\tilde{s}\rangle$ acts on $\langle\tilde{q},\tilde{r}\rangle$ as follows:

$$\tilde{p}\tilde{q}\tilde{p} = \tilde{q},\ \tilde{p}\tilde{r}\tilde{p} = \tilde{t} = \tilde{r}\tilde{q},$$
$$\tilde{s}\tilde{q}\tilde{s} = \tilde{t} = \tilde{r}\tilde{q},\ \ \tilde{s}\tilde{r}\tilde{s} = \tilde{r}.$$

Hence $A(c) \cong S_4$ and similarly, $A(b) \cong S_4$.

Thus in this case $X \cong Y \cong S_4$ and $H \cong D_4$.

Case $s = 5$. Let α be in the center of $A(c,d)$, see Figure 3.

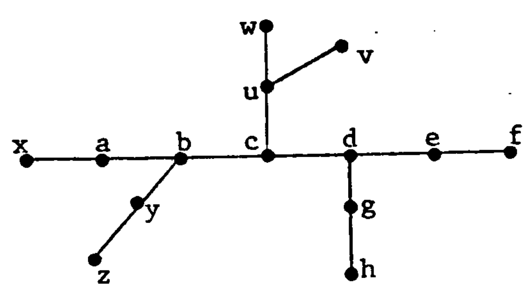


Figure 3

We claim that α fixes also b and e. Indeed, if $\alpha(b) = u$ and say $\alpha(a) = v$ then let $\beta \neq 1$ be in $A(x,a,b,c,d)$. Since α and β commute it follows that β also fixes $u = \alpha(b)$ and $v = \alpha(a)$. Thus β fixes a 5-arc (x,a,b,c,u,v) and so $\beta = 1$, a contradiction.

Thus $\alpha \in A(b,c,d,e)$ and so the center of $A(c,d)$ has order 2 or 4.

Assume that this center has order 2 and let $\alpha \neq 1$ be its generator. Then $\alpha(b) = b$, $\alpha(e) = e$ but α must move either a or f, say it moves a. Thus $\alpha(a) = y$. Let $\beta \neq 1$ be in $A(x,a,b,c,d)$. If $\alpha(x) = z$ then since $\alpha\beta = \beta\alpha$, it follows that β also fixes y and z. Thus β fixes all vertices at distance 2 from b and necessarily moves all vertices at distance 3 from b. Thus β is uniquely determined by the vertex b and we shall write $\beta = \tilde{b}$. Let us say that $b \in V^+$. Then for every vertex $t \in V^+$ we have the corresponding involution $\tilde{t}$.

Let $\gamma \neq 1$ be in $A(a,b,c,d,e)$. We claim that $\gamma(v) = w$. Otherwise γ fixes all vertices at distance 2 from c and moves all vertices at distance 3. This implies that γ lies in the center of $A(c)$ and so the center of $A(c,d)$ contains $\langle\beta,\gamma\rangle$ which has order 4, a contradiction. Thus we have proved that $\gamma(v) = w$.

It follows that $A(c)$ contains 3 involutions which fix a 4-arc with c as midpoint. We say that any one of these three involutions is associated to c and will denote one of them by $\hat{c}$. The analogous claim is valid for every vertex $t \in V^-$.

Choose $\hat{c} = \gamma$ and $\hat{e}$ so that $\hat{e}(c) = c$. Then $\hat{e}(u) = b$ and so $\hat{e}$ and $\hat{c}$ do not commute. But $(\hat{e}\hat{c})^2$ fixes b,c,d,e,f and so $(\hat{e}\hat{c})^2 = \tilde{d}$. Therefore $\langle\hat{e},\hat{c}\rangle = A(c,d,e) \cong D_4$.

Since $\hat{e}(b) = u$ we have $\hat{e}\tilde{b}\hat{e} = \tilde{u}$ and so $(\hat{e}\tilde{b})^2 = \widetilde{ub}$ is an element of order 2. On the other hand $\tilde{b}(e) = g$, so $\tilde{b}\hat{e}\tilde{b} = \hat{g}$ and $(\hat{e}\tilde{b})^2 = \hat{e}\hat{g}$. But $\langle\hat{e},\hat{g}\rangle \cong D_4$ and so $\hat{e}\hat{g}$ and $(\hat{e}\tilde{b})^2$ are elements of order 4. This is a contradiction.

Thus we have proved that the center of $A(c,d)$ must be of order 4, and so it coincides with $A(b,c,d,e)$. Thus it is an elementary abelian group of order 4. Let α be the generator of $A(a,b,c,d,e)$. Since α belongs to the center of $A(c,d)$ it follows that $\alpha(u) = u$. Thus α fixes all vertices at distance 2 from c and moves all those at distance 3. Thus we may write $\alpha = \tilde{c}$ since it is uniquely determined by c. Similarly, we have $\tilde{d}$ which generates $A(b,c,d,e,f)$. Thus for every vertex t there is a unique involution $\tilde{t}$ associated to it in this manner.

Now we find that

$$A(a,b,c,d) = \langle\tilde{b},\tilde{c}\rangle \cong C_2 \times C_2,$$

$$A(b,c,d) = \langle\tilde{b},\tilde{c},\tilde{d}\rangle \cong C_2 \times C_2 \times C_2.$$

Since $\tilde{b}(e) = g$ we have $\widetilde{\tilde{b}\tilde{e}\tilde{b}} = \tilde{g}$ and so $\widetilde{be} \neq \widetilde{eb}$. But $(\widetilde{be})^2$ fixes b,c,d,e and so belongs to $\langle\tilde{c},\tilde{d}\rangle$. Since $(\widetilde{be})^2 = \widetilde{bu} = \widetilde{ge}$ it follows that it moves both a and f and so we must have $(\widetilde{be})^2 = \widetilde{cd}$.

Thus

$$A(c,d) = \langle\tilde{b},\tilde{c},\tilde{d},\tilde{e}\rangle = \langle\tilde{b},\tilde{e}\rangle \times \langle\tilde{c}\rangle$$

$$\cong D_4 \times C_2.$$

Now $\widetilde{ae}$ permutes cyclically the vertices b,d,u and so $(\widetilde{ae})^3 \in A(b,c,d)$. Thus $\widetilde{ae}$ has order 3 or 6. The same argument as in the previous cases shows that the order must be 3. Hence $\langle\tilde{a},\tilde{e}\rangle \cong D_3$. We claim that $\langle\tilde{a},\tilde{e}\rangle$ normalizes the four-group $\langle\widetilde{bc},\widetilde{cd}\rangle$. Indeed we have

$$\tilde{a}(\widetilde{bc})\tilde{a} = \widetilde{bc},$$
$$\tilde{a}(\widetilde{cd})\tilde{a} = \tilde{c}(\widetilde{ad})^2\tilde{d} = \widetilde{cbcd} = \widetilde{bd},$$

and so $\tilde{a}$ normalizes this four-group. Similarly, $\tilde{e}$ normalizes this four-group. Consequently

$$\langle\tilde{a},\widetilde{bc},\widetilde{cd},\tilde{e}\rangle \cong S_4 \quad \text{(the symmetric group)}$$

and

$$A(c) = \langle\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e}\rangle = \langle\tilde{a},\widetilde{bc},\widetilde{cd},\tilde{e}\rangle \times \langle\tilde{c}\rangle$$
$$\cong S_4 \times C_2.$$

Similarly, $A(d) \cong S_4 \times C_2$.

Thus we have in this case

$$X \cong Y \cong S_4 \times C_2, \quad H \cong D_4 \times C_2.$$

The case $s = 7$ will be considered in the next section.

8. END OF PROOF OF THEOREM 11

It remains to consider the case $s = 7$. We may assume that our amalgam is associated with a pair (G,A) where G is a connected bipartite cubic graph and $A \le \mathrm{Aut}(G)$ is locally 7-regular. If (V^+,V^-) is the underlying partition of $V = V(G)$ then A necessarily preserves V^+ and V^- since A cannot be 7-regular.

It follows from local 7-regularity of A that a fixer in A of a vertex in G has order $3 \cdot 2^6$, and the fixer of an i-arc $(1 \le i \le 7)$ has order 2^{7-i}. The fixers of some vertices and arcs in Figure 4 are sketched in Figure 5.

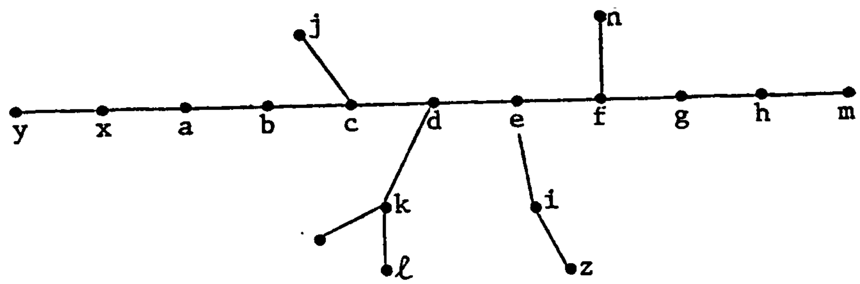


Figure 4

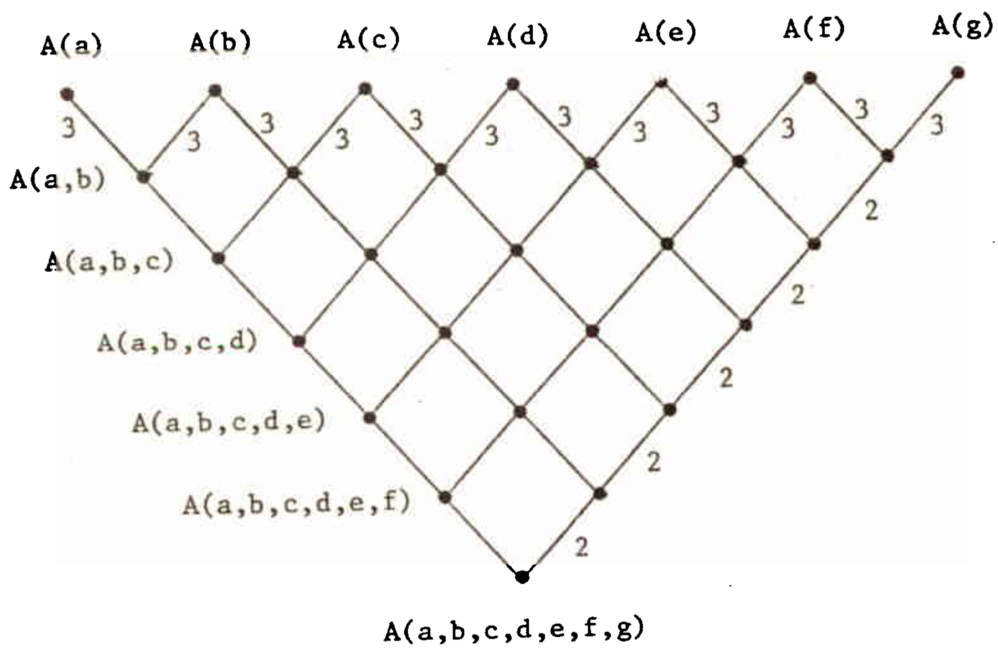


Figure 5

Assume first that for every vertex $u \in V$ there exists $\alpha \neq 1$ in A such that α fixes all vertices at distance ≤ 3 from u. Then local 7-regularity of A implies that α moves every vertex at distance 4 from u. Clearly α is uniquely determined by the vertex u and we shall write $\alpha = \tilde{u}$. If $\beta \in A$ and $\beta(u) = v$ then it is clear that $\beta\tilde{u}\beta^{-1} = \tilde{v}$, by using the above geometric characterization of the involutions $\tilde{u}$. Then (see Figure 4) we have

$$A(a,b,c,d,e,f,g) = \langle\tilde{d}\rangle \cong C_2,$$

$$A(b,c,d,e,f,g) = \langle\tilde{d},\tilde{e}\rangle \cong C_2 \times C_2,$$

$$A(b,c,d,e,f) = \langle\tilde{c},\tilde{d},\tilde{e}\rangle \cong C_2 \times C_2 \times C_2,$$

$$A(c,d,e,f) = \langle\tilde{c},\tilde{d},\tilde{e},\tilde{f}\rangle \cong C_2 \times C_2 \times C_2 \times C_2,$$

because, for instance, $\tilde{c}(f) = f$ implies that $\tilde{c}\tilde{f}\tilde{c} = \tilde{f}$, i.e., $\tilde{c}$ and $\tilde{f}$ commute.

Since $\tilde{b}(f) = i$ and $\tilde{f}(b) = j$ it follows that $\tilde{b}\tilde{f}\tilde{b} = \tilde{i}$ and $\tilde{f}\tilde{b}\tilde{f} = \tilde{j}$. Hence we have $(\tilde{b}\tilde{f})^2 = \tilde{i}\tilde{f} = \tilde{b}\tilde{j}$ which shows that $(\tilde{b}\tilde{f})^2$ fixes all the vertices a,b,c,d,e,f,g and moves the vertex h. This implies that $(\tilde{b}\tilde{f})^2 = \tilde{d}$. On the other hand $\tilde{b}\tilde{f} \in A(c,d,k,\ell)$ and $A(c,d,k,\ell) = \langle\tilde{c},\tilde{d},\tilde{k},\tilde{\ell}\rangle$ is elementary abelian group of order 16. This forces that $(\tilde{b}\tilde{f})^2 = 1$ and we have a contradiction.

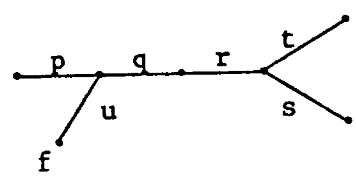
Thus we have proved that there exists a vertex $u \in V$ such that if $\alpha \in A$ fixes all vertices at distance ≤ 3 from u then $\alpha = 1$. We may assume that $u \in V^-$, and hence every vertex in V^- has this property.

Now we claim that $A(a,b,c,d)$ is non-abelian. Otherwise the fixer of every 3-arc would be abelian. Since $A(d)$ is generated by the abelian subgroups $A(a,b,c,d)$, $A(b,c,d,e)$, $A(c,d,e,f)$, and $A(d,e,f,g)$ and each of them contains the non-trivial element α of $A(a,b,c,d,e,f,g)$, it follows that α lies in the center of $A(d)$. But $A(d)$ acts transitively on 6-arcs having d as midpoint and α fixes one of them. Since α is central in $A(d)$ it follows that α fixes all of these 6-arcs and hence α fixes all vertices at distance ≤ 3 from d. But $\alpha \neq 1$ and since d is arbitrary this contradicts the fact established above. Thus we have proved that $A(a,b,c,d)$ is non-abelian.

Let α belong to the center of $A(d,e)$. Then we claim that α fixes every vertex at distance ≤ 2 from the edge $\{d,e\}$. Indeed, assume that, say, $\alpha(b) = j$. Let (x,y,a,b,c,d,e) be a 6-arc and let $\beta \neq 1$ be an involution in A fixing this 6-arc. Since $\alpha\beta = \beta\alpha$ and $\alpha(b) = j$, it follows that β fixes the 8-arc $(x,y,a,b,c,j,\alpha(a),\alpha(y),\alpha(x))$. This is a contradiction since $\beta \neq 1$ and A is locally 7-regular. Thus, α must fix every vertex at distance ≤ 2 from $\{d,e\}$. Hence the center of $A(d,e)$ is contained in the group $A(b,c,d,e,f,g)$. This latter group has order 4 and is elementary abelian since it is generated by involutions.

We claim that the center of $A(d,e)$ has order 2. Otherwise the center of $A(d,e)$ would coincide with $A(b,c,d,e,f,g)$. Let $\alpha \neq 1$ be the involution fixing the vertices a,b,c,d,e,f,g. Since α belongs to the center of $A(d,e)$, an argument which we used earlier shows that α fixes all vertices at distance ≤ 3 from d. Similarly, if $\beta \neq 1$ is the involution fixing b,c,d,e,f,g,h then it fixes all vertices at distance ≤ 3 from e. This contradicts the fact proved above about vertices in V^-. Consequently, we have proved that the center of $A(d,e)$ has order 2.

Let $\alpha \ne 1$ be the central element of $A(d,e)$. Assume that α moves every vertex at distance 3 from the edge $p = \{d,e\}$. Then α is uniquely determined by this edge and we can write $\alpha = \tilde{p}$. A similar statement is then valid for every edge of G since A is transitive on the edges of G. If $\beta \in A$ and $\beta(p) = q$ then we have $\beta\tilde{p}\beta^{-1} = \tilde{q}$, because of the geometric description of the involutions $\tilde{p}$. Since (see Figure 6)



<u>Figure 6</u>

$\widetilde{pu}$ fixes all vertices at distance ≤ 2 from the edge q and $\widetilde{pu} \ne 1$, it follows that $\widetilde{pu} = \tilde{q}$. Now, we have $\tilde{p}(t) = s$ and $\tilde{t}(p) = u$ and consequently

$$\widetilde{ptp} = \tilde{s} \quad \text{and} \quad \widetilde{tpt} = \tilde{u}.$$

Thus $(\widetilde{pt})^2 = \widetilde{st} = \tilde{r}$ and $(\widetilde{pt})^2 = \widetilde{pu} = \tilde{q}$, and we have a contradiction since $\tilde{r} \ne \tilde{q}$.

Hence we have proved that α must fix at least one vertex at distance 3 from $\{d,e\}$. Say, we have $\alpha(h) = h$. Then α is the involution in A which fixes the 6-arc (b,c,d,e,f,g,h). Since α is central in $A(d,e)$, the usual argument shows that α fixes every vertex at distance ≤ 3 from e. Thus α is uniquely determined by the vertex e and we shall write $\alpha = \tilde{e}$. Consequently, we have $e \in V^+$ and for every $u \in V^+$ there is an associated involution $\tilde{u}$. We repeat that every vertex at distance ≤ 3 from u is fixed by $\tilde{u}$ and every vertex at distance 4 from u is necessarily moved by $\tilde{u}$.

For each 6-arc S let $\tilde{S}$ be the unique involution in A which fixes S. If the midpoint $S(3) = u$ is in V^+ then we know that $\tilde{S} = \tilde{u}$.

Now assume that $S(3) = u \in V^{-1}$. Then we claim that $\tilde{S}$ has the form described on Figure 7 where the double arrows indicate the vertices which are interchanged by $\tilde{S}$.

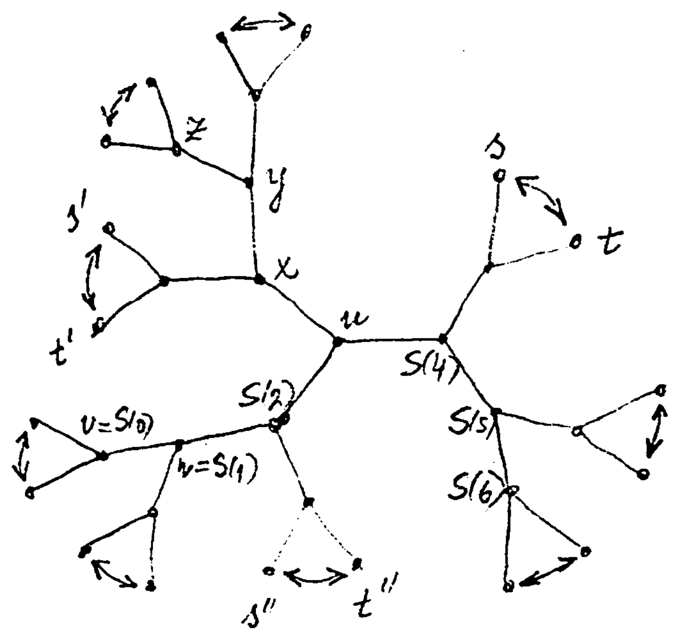


Figure 7

Indeed, let $v = S(0)$, $w = S(1)$. Then, say, $\tilde{v}(S(4)) = x$, $\tilde{v}(S(5)) = y$, $\tilde{v}(S(6)) = z$. Now, $\tilde{v}$ belongs to the center of $A(v,w)$ and $\tilde{S} \in A(v,w)$. It follows that $\tilde{v}\tilde{S} = \tilde{S}\tilde{v}$ and consequently $\tilde{S}$ also fixes x,y,z. Since $u \in V^-$, $\tilde{S}$ cannot fix all vertices at distance 3 from u. Say $\tilde{S}$ interchanges s and t. If $s' = \tilde{v}(s)$ and $t' = \tilde{v}(t)$ then $\tilde{S}$ must also interchange s' and t'. Let $\tilde{z}(s) = s''$ and $\tilde{z}(t) = t''$. Since $\tilde{z}\tilde{S} = \tilde{S}\tilde{z}$, it follows that $\tilde{S}$ also interchanges s'' and t''. The remaining pairs of vertices indicated by arrows on Figure 7 must be interchanged by $\tilde{S}$ because A is locally 7-regular and $\tilde{S} \neq 1$. Thus our claim about $\tilde{S}$ is proved.

It is clear from Figure 7 that there are 4 different involutions $\tilde{S}$ such that $S(3) = u$ where $u \in V^-$.

From now on we shall use the notation from Figure 4. The vertices a,c,e,g are in V^+ and hence we have canonical involutions $\tilde{a},\tilde{c},\tilde{e},\tilde{g}$. Let $\hat{d}$ be the involution fixing vertices a,b,c,d,e,f,g.

We have

$$A(a,b,c,d,e,f,g) = \langle \hat{d} \rangle \cong C_2,$$

$$A(b,c,d,e,f,g,h) = \langle \tilde{e} \rangle \cong C_2.$$

Since $\hat{d}(e) = e$ we have $\hat{d}\tilde{e}\hat{d} = \tilde{e}$, i.e., $\hat{d}$ and $\tilde{e}$ commute. Hence

$$A(b,c,d,e,f) = \langle \hat{d},\tilde{e} \rangle \cong C_2 \times C_2.$$

Similarly,

$$A(b,c,d,e,f) = \langle \tilde{c},\hat{d},\tilde{e} \rangle \cong C_2 \times C_2 \times C_2,$$

since, for instance, $\tilde{c}(e) = e$ implies that $\tilde{c}$ and $\tilde{e}$ commute.

Let $\hat{f} \in A$ be the involution fixing c,d,e,f,g,h,m. Since $\hat{d}\hat{f}$ fixes the vertices c,d,e,f,g, we have that $(\hat{d}\hat{f})^2$ fixes also b and h. Therefore $(\hat{d}\hat{f})^2$ is either 1 or $\tilde{e}$. But $\hat{f}(b) = j$ implies that $\hat{f}\hat{d}\hat{f}$ moves a and so $\hat{f}\hat{d}\hat{f} \neq \hat{d}$, i.e., $(\hat{d}\hat{f})^2 \neq 1$. This proves that $(\hat{d}\hat{f})^2 = \tilde{e}$. Hence

$$A(c,d,e,f,g) = \langle\hat{d},\tilde{e},\hat{f}\rangle = \langle\hat{d},\hat{f}\rangle \cong D_4.$$

Since $\hat{f}(c) = c$, the involutions $\tilde{c}$ and $\hat{f}$ commute. Thus

$$\begin{aligned} A(c,d,e,f) &= \langle\tilde{c},\hat{d},\tilde{e},\hat{f}\rangle = \langle\hat{d},\tilde{e},\hat{f}\rangle \times \langle\tilde{c}\rangle \\ &\cong D_4 \times C_2. \end{aligned}$$

Since $\tilde{c}(g) = n$ and $\tilde{g}(c) = k$ we have $\tilde{c}\tilde{g}\tilde{c} = \tilde{n}$ and $\tilde{g}\tilde{c}\tilde{g} = \tilde{k}$. Thus

$$(\tilde{c}\tilde{g})^2 = \tilde{n}\tilde{g} = \tilde{c}\tilde{k}$$

which shows that $(\tilde{c}\tilde{g})^2$ fixes the vertices b,c,d,e,f,g,h. Since also $\tilde{c}\tilde{g}\tilde{c} = \tilde{n} \neq \tilde{g}$ we have $(\tilde{c}\tilde{g})^2 \neq 1$ and consequently $(\tilde{c}\tilde{g})^2 = \tilde{e}$. Thus $\langle\tilde{c},\tilde{g}\rangle \cong D_4$ and

$$A(d,e,f) = \langle\tilde{c},\hat{d},\tilde{e},\hat{f},\tilde{g}\rangle \cong D_4 \odot D_4$$

where $\odot$ denotes central product of groups (see [9, p.]).

Let $\hat{b} \in A$ be the involution fixing the vertices y,x,a,b,c,d,e. We have $\hat{b}(f) = i$ and $\hat{f}(b) = j$, and put $\hat{i} = \hat{b}\hat{f}\hat{b}$, $\hat{j} = \hat{f}\hat{b}\hat{f}$. Since $\hat{i}(g) = n$ and $\hat{j}(a) \neq a$, it follows that the element

$$(\hat{b}\hat{f})^2 = \hat{i}\hat{f} = \hat{b}\hat{j}$$

fixes b,c,d,e,f and moves a and g. Hence $(\hat{b}\hat{f})^2\tilde{c}\tilde{e}$ fixes the vertices a,b,c,d,e,f,g and so it is equal to 1 or $\hat{d}$. But $\hat{b}$ and $\hat{f}$ fix k

and move ℓ. Hence $\hat{b}\hat{f}$ fixes k and ℓ and consequently $(\hat{b}\hat{f})^2$ fixes all vertices at distance 2 from k. The same is true for $\tilde{c}\tilde{e}$ and consequently for $(\hat{b}\hat{f})^2\tilde{c}\tilde{e}$. This fact and the description of elements $\tilde{S}$ on Figure 7 show that $(\hat{b}\hat{f})^2\tilde{c}\tilde{e} \neq \hat{d}$. Therefore we have $(\hat{b}\hat{f})^2 = \tilde{c}\tilde{e}$. Hence

$$A(c,d,e) = \langle\hat{b},\tilde{c},\hat{d},\tilde{e},\hat{f}\rangle$$

is a group of order 32. Omitting tildes and hats we have the following defining relations for this group:

$$b^2 = c^2 = d^2 = e^2 = f^2 = 1,$$

$$c \text{ and } e \text{ are central,}$$

$$(bd)^2 = c, \quad (df)^2 = e, \quad (bf)^2 = ce.$$

We have, say, $\hat{b}(g) = z$ and $\tilde{g}(b) = \ell$. Thus $\hat{b}\tilde{g}\hat{b} = \tilde{z}$ and we put $\hat{\ell} = \tilde{g}\hat{b}\tilde{g}$. Then $\hat{\ell}$ fixes k,d,e and we have

$$(\hat{b}\tilde{g})^2 = \tilde{z}\tilde{g} = \hat{b}\hat{\ell}.$$

It follows from here that $(\hat{b}\tilde{g})^2$ fixes the vertices c,d,k,e,f,i and moves the vertices g,z,ℓ,b. The element $(\hat{b}\tilde{g})^2\tilde{c}\hat{f}$ fixes the vertices b,c,d,e,f,g and moves z. Therefore, it belongs to $\langle\hat{d},\tilde{e}\rangle$ and since it moves z, we conclude that

$$(\hat{b}\tilde{g})^2\tilde{c}\hat{f}\hat{d} \in \langle\tilde{e}\rangle.$$

By replacing $\hat{f}$ by $\hat{f}\tilde{e}$ (if necessary) we may assume that

$$(\hat{b}\tilde{g})^2 = \tilde{c}\hat{d}\tilde{e}\hat{f}.$$

Thus we have

$$A(d,e) = \langle\hat{b},\tilde{c},\hat{d},\tilde{e},\hat{f},\tilde{g}\rangle$$

with defining relations the same as for A(c,d,e) plus the following:

$$g^2 = (gf)^2 = (ge)^2 = (gd)^2 = 1, \quad (gc)^2 = e, \quad (bg)^2 = cdef.$$

The element $\widetilde{a}\widetilde{g}$ permutes cyclically the vertices c,e,k. Thus $(\widetilde{a}\widetilde{g})^3$ is in A(c,d,e) and so the order of $\widetilde{a}\widetilde{g}$ is 3,6, or 12. But $(\widetilde{a}\widetilde{g})^2 = \widetilde{a}\widetilde{u}$ where $u = \widetilde{g}(a)$ and so it is conjugate in A to $\widetilde{a}\widetilde{g}$. Therefore ag has order 3. Hence

$$A(d) = \langle \widetilde{a},\hat{b},\widetilde{c},\hat{d},\widetilde{e},\hat{f},\widetilde{g} \rangle$$

with defining relations the same as those for A(d,e) plus the following:

$$a^2 = (ab)^2 = (ac)^2 = (ad)^2 = (ag)^3 = 1,$$
$$(ae)^2 = c, \quad (af)^2 = bcde.$$

Let $\hat{h} \in A$ be an involution fixing a 6-arc (e,f,g,h,m,...). Then we have again $(\hat{b}\hat{h})^3 = 1$, by a similar argument. Hence

$$A(e) = \langle \hat{b},\widetilde{c},\hat{d},\widetilde{e},\hat{f},\widetilde{g},\hat{h} \rangle$$

with defining relations the same as those of A(d,e) plus the following:

$$h^2 = (he)^2 = (hg)^2 = (hb)^3 = 1,$$
$$(dh)^2 = eg, \quad (fh)^2 = g, \quad (ch)^2 = defg.$$

Only the last relation needs justification. We have

$$(\widetilde{c}\hat{h})^2\widetilde{g}\hat{f}\hat{d} \in \langle \widetilde{e} \rangle,$$

proved in the same way as $(\hat{b}\widetilde{g})^2\widetilde{c}\hat{f}\hat{d} \in \langle \widetilde{e} \rangle$. Thus either $(\widetilde{c}\hat{h})^2 = \hat{d}\widetilde{e}\hat{f}\widetilde{g}$ or $(\widetilde{c}\hat{h})^2 = \hat{d}\hat{f}\widetilde{g}$. In the second case we need only replace $\hat{h}$ by $\hat{h}\widetilde{g} = \hat{h}'$. Indeed, we then have

$$(\widetilde{c}\hat{h}')^2 = \widetilde{c}\hat{h}\widetilde{g}\widetilde{c}\hat{h}\widetilde{g} = \widetilde{c}\hat{h}\widetilde{c}\widetilde{g}\widetilde{e}\hat{h}\widetilde{g}$$
$$= (\widetilde{c}\hat{h})^2\widetilde{e} = \hat{d}\widetilde{e}\hat{f}\widetilde{g}.$$

This completes the proof in the case $s = 7$.

We summarize the result by giving the amalgam $(X,Y;H)$ in terms of generators and defining relations.

Thus

$$X = \langle a,b,c,d,e,f,g \rangle,$$

$$Y = \langle b,c,d,e,f,g,h \rangle,$$

$$H = X \cap Y = \langle b,c,d,e,f,g \rangle,$$

$$a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = g^2 = h^2 = 1,$$

$$(ab)^2 = (bc)^2 = (cd)^2 = (de)^2 = (ef)^2 = (fg)^2 = (gh)^2 = 1,$$

$$(ac)^2 = (ce)^2 = (eg)^2 = 1,$$

$$(bd)^2 = c, \quad (df)^2 = e, \quad (fh)^2 = g,$$

$$(ad)^2 = (be)^2 = (cf)^2 = (dg)^2 = (eh)^2 = 1,$$

$$(ae)^2 = c, \quad (cg)^2 = e,$$

$$(bf)^2 = ce, \quad (dh)^2 = eg,$$

$$(af)^2 = bcde, \quad (bg)^2 = cdef, \quad (ch)^2 = defg,$$

$$(ag)^3 = (bh)^3 = 1.$$

9. SOME REMARKS ABOUT AMALGAMS

Let $(X,Y;H)$ be a simple amalgam and let $(X',Y';H')$ be a transitive subamalgam of it (see section 2 for the definition). Then we claim that the only subgroup of H which is normalized by both X' and Y' is $\{1\}$. In particular, the subamalgam $(X',Y';H')$ is simple.

The proof is easy: By Theorem 9 we may assume that $(X,Y;H)$ is the amalgam associated to (G,A) where G is a connected bipartite graph, with underlying partition (V^+,V^-) of $V = V(G)$ and A is a locally 1-transitive subgroup of $Aut(G)$ preserving V^+ and V^-. Say $X = A(a)$ and $Y = A(b)$ where $\{a,b\}$ is an edge of G, $a \in V^+$, $b \in V^-$. Let A' be the subgroup of A generated by X' and Y'. By hypothesis A' is transitive on V^+ and on V^-. If $N \leq H$ is normalized by both X' and Y' then N fixes a and b and also every vertex $z(a)$ and $z(b)$ for $z \in A'$. Since V^+ and V^- are two orbits of A' and $a \in V^+$, $b \in V^-$, it follows that N fixes every vertex of G. This implies that $N = \{1\}$ because $A \leq Aut(G)$.

We leave to the reader to find a purely group-theoretical proof of this result, which is not difficult.

Second remark is about finite simple amalgams $(X,Y;H)$ of degree (d^+,d^-). We claim that if this amalgam is proper and a prime p divides the order of H then $p < d^+$ or $p < d^-$. Indeed, then we have (G,A) as in the first remark.

If $\alpha \in H$ is an element of order p and if $p \geq d^+$ and $p \geq d^-$ then since α fixes the vertices a and b these inequalities imply that α must fix all neighbours of a and b, etc. Thus $\alpha = 1$ and we have a contradiction.

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