Using Simple Physical Models for Image Segmentation

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What is Image Segmentation?

Input

Output

Three Talks in One

- Image Segmentation Experimental
- Image Segmentation Theory
- Solving needed linear systems

What is Hard for Computers!

The CAPTCHA Project

In the spaces below, type three (3) different English words appearing in the picture above.

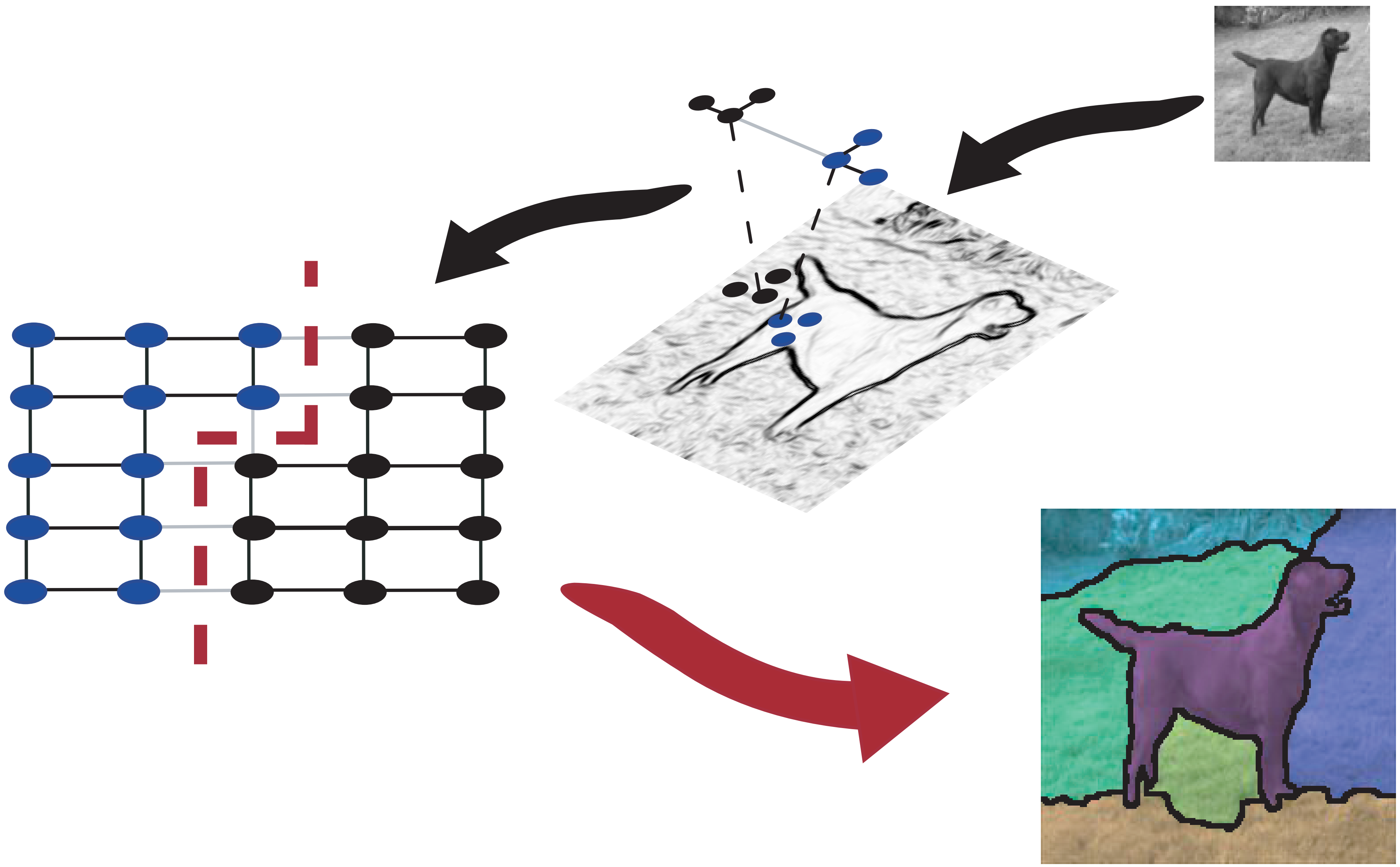
Image Processing is every where

- ✓ Medical Image Analysis
- ✓ Matting & Manipulation
- ✓ Data Mining & Information Retrieval
- ✓ Intelligent Surveillance Systems

CS Reduction

- Convert the image segmentation problem into a well studied computer science problem.
- Hopefully, use an off-the-shelf solution to the CS problem.
- First attempt, Shi and Malik (2000)

Image Segmentation as Graph Partitioning



Basic Approach

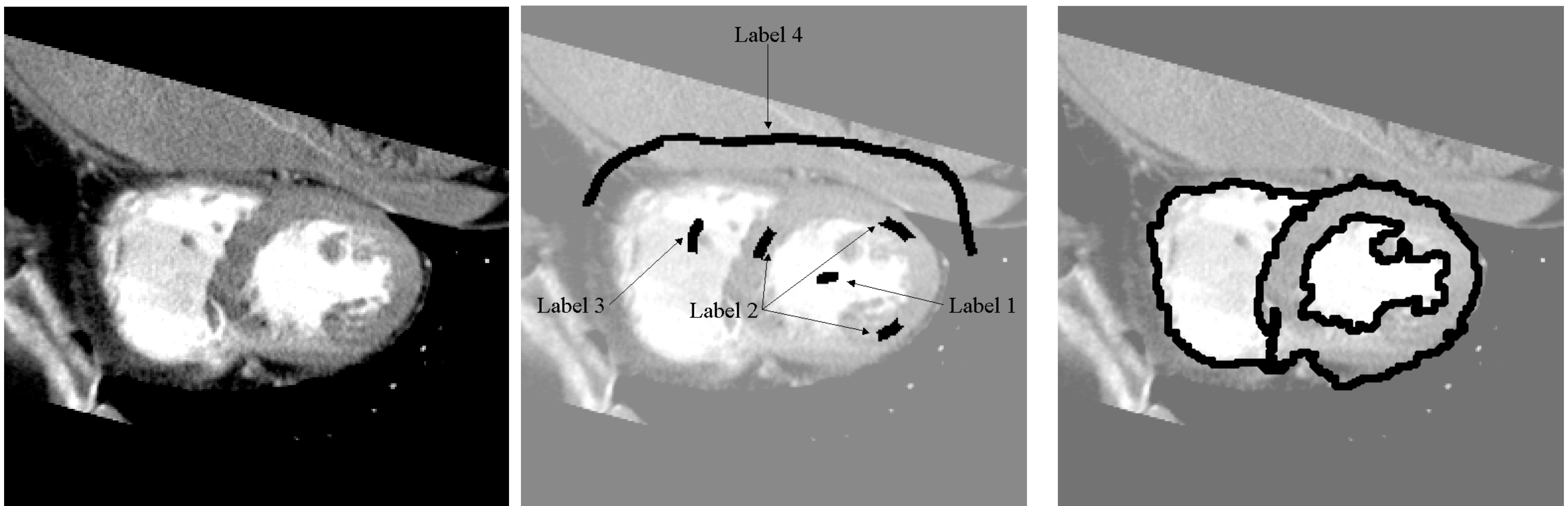
Generate an affinity graph

- Each pixel a vertex
- Neighboring pixels are connected with an edge
- The weight w_{ij} corresponds to their similarity

Ways to View edges

- Max-flow min-cut models
  - Each edge is a "pipe" that can carry a flow up to $\mathbf{w_{ij}}$
  - Pick a source and sink and find mincut
- Electrical conductor models
  - Each edge is a conductor of size $\mathbf{w_{ij}}$
  - Set voltage on some nodes to +1 and some to -1

Siemens Assisted Segmentation 2005



(a) Original (b) Seeds indicating four objects (c) Resulting segmentation

User assisted segmentation of the heart

no prior knowledge of hearts

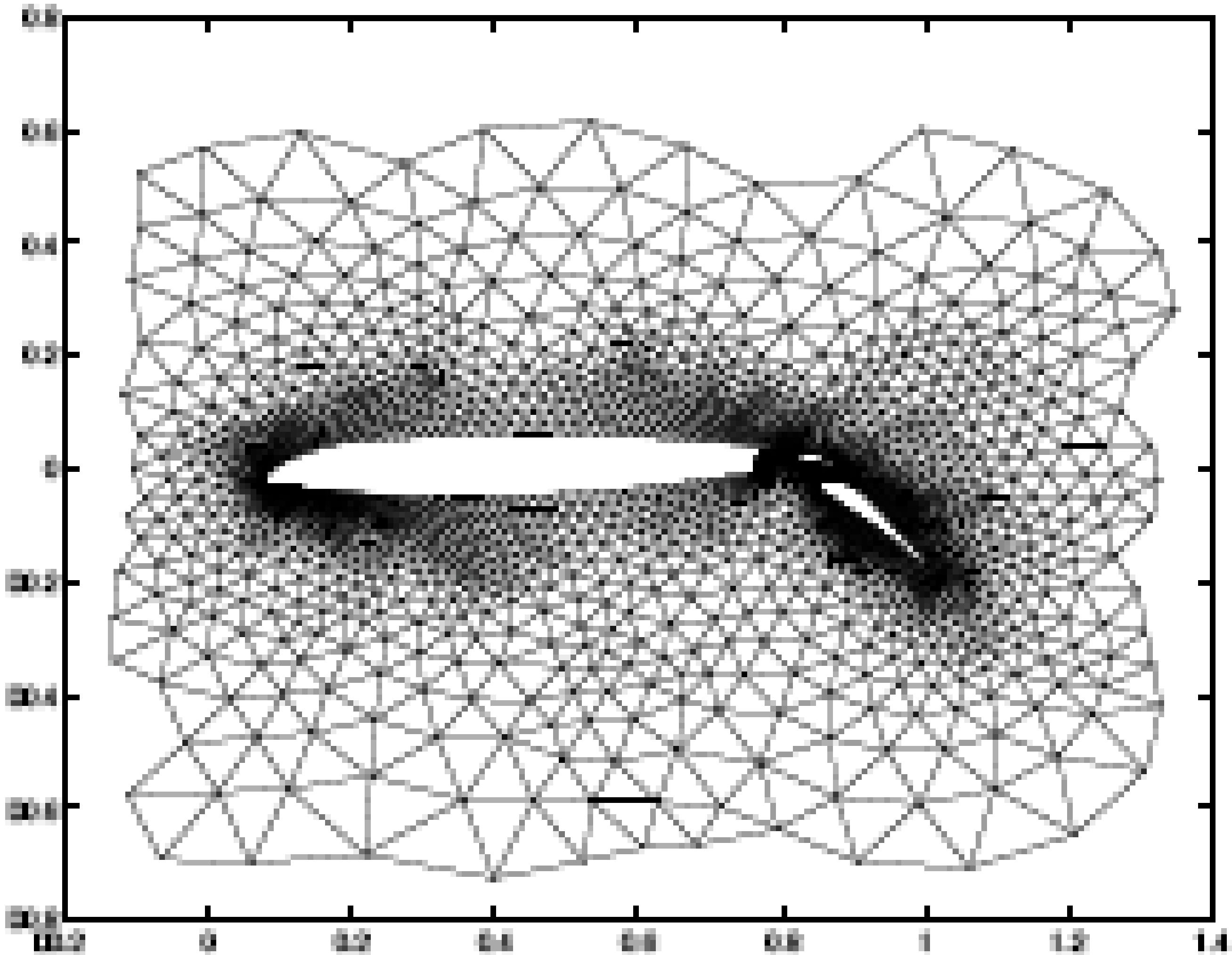
More Ways to View edges

- Shortest Path Models
  - The distance between neighbors is $\mathbf{1/w_{ij}}$
- Random Walk Models
  - The probability we walk on an edge is proportional to $\mathbf{w_{ij}}$
  - Distance between node is the expect commute time

Even More Ways to View edges

- Spring Models (This talk)
  - The spring constant is $\mathbf{w_{ij}}$
  - Separate based on modes of vibration (eigenvectors).
- Classic Spring Solution.
  - Compute a few low frequency eigenvectors say 2.
  - Map the vertices into 2D using the eigenvectors.
  - Apply a geometric cut the graph.

Airfoil Graph and it Spectral Embedding



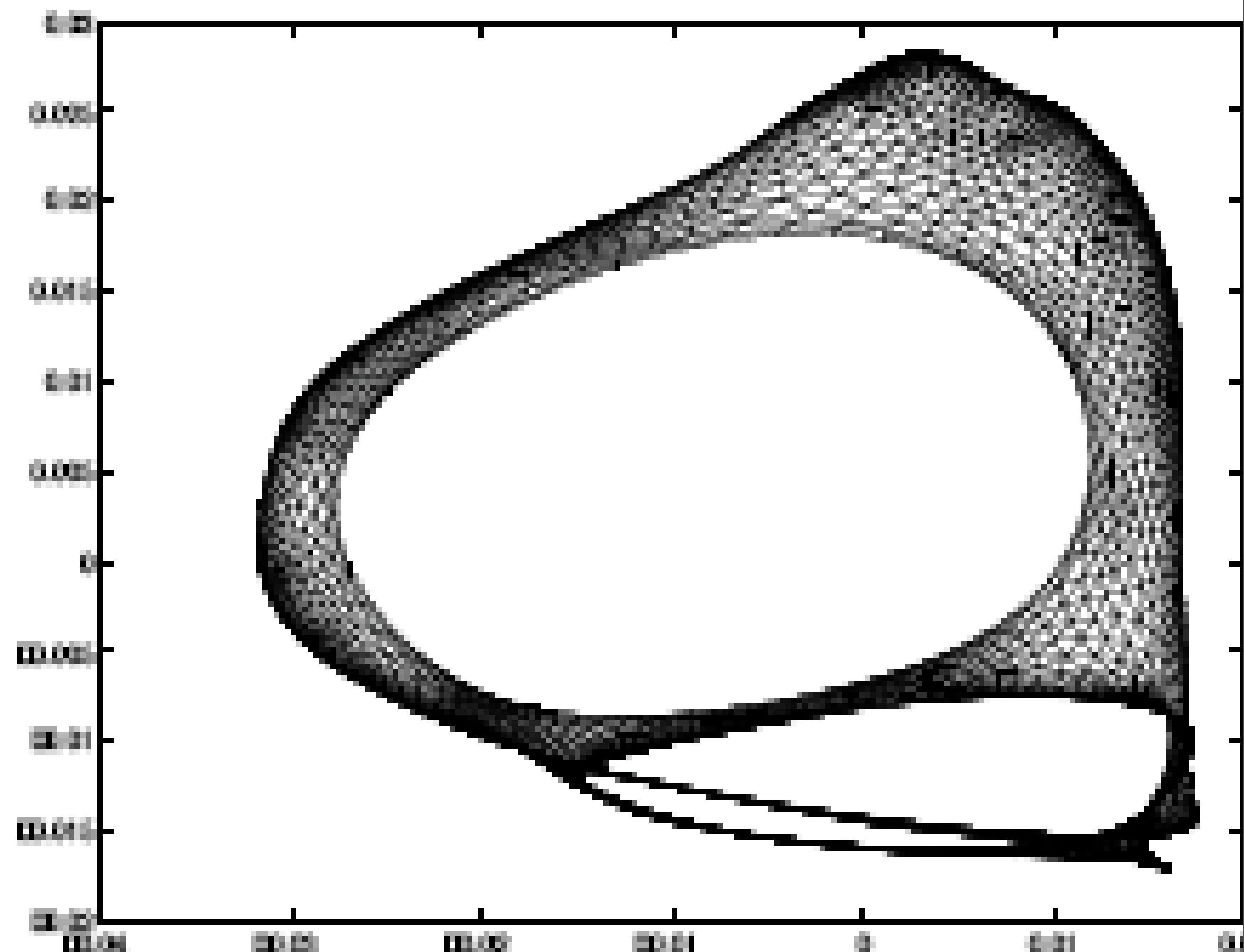


Image Segmentation as Graph Partitioning

Goal: segment into 4 pieces

Output of the Classic Spring Model

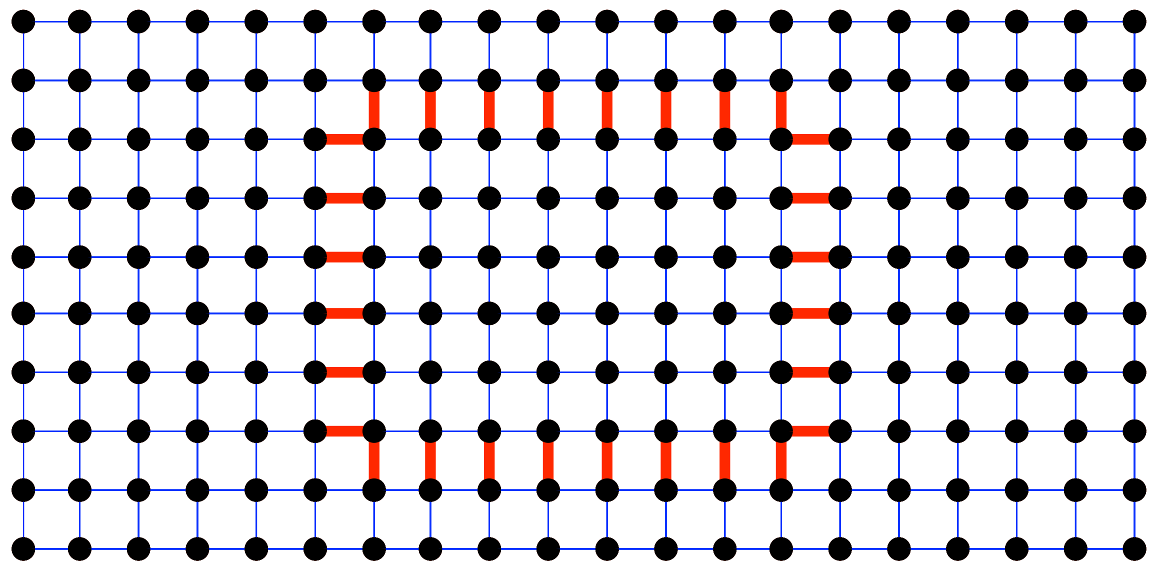
Spring model using 4 eigenvectors

Output for Our Spectral Rounding Algorithm

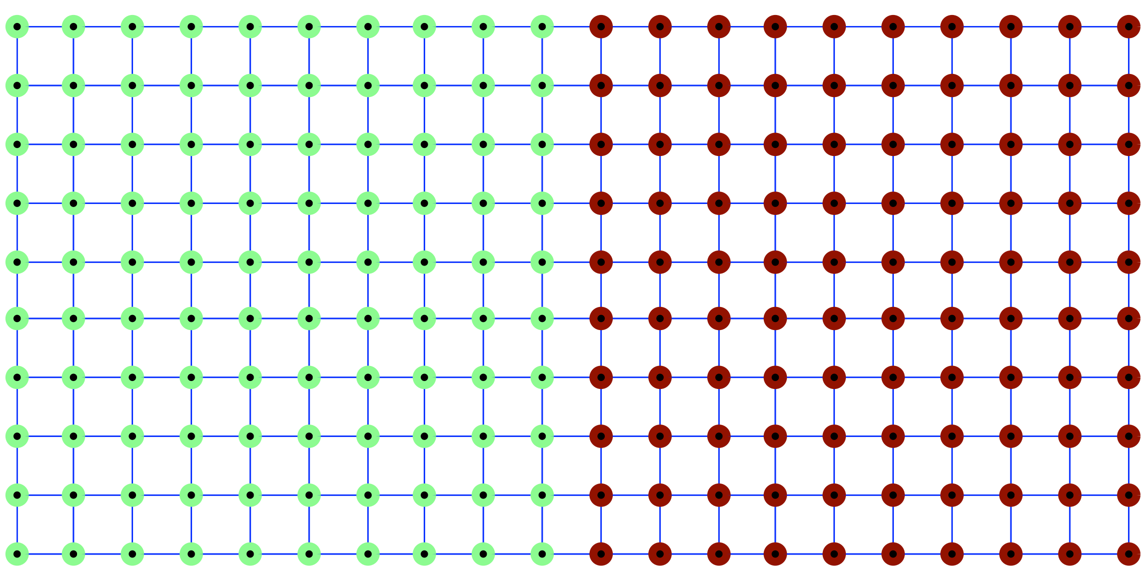
What is new

- **Spectral Rounding :**
  - A better method to use eigenvectors for graph partitioning.
- **Fast Planar Solvers :**
  - Optimal time linear solvers for planar systems.

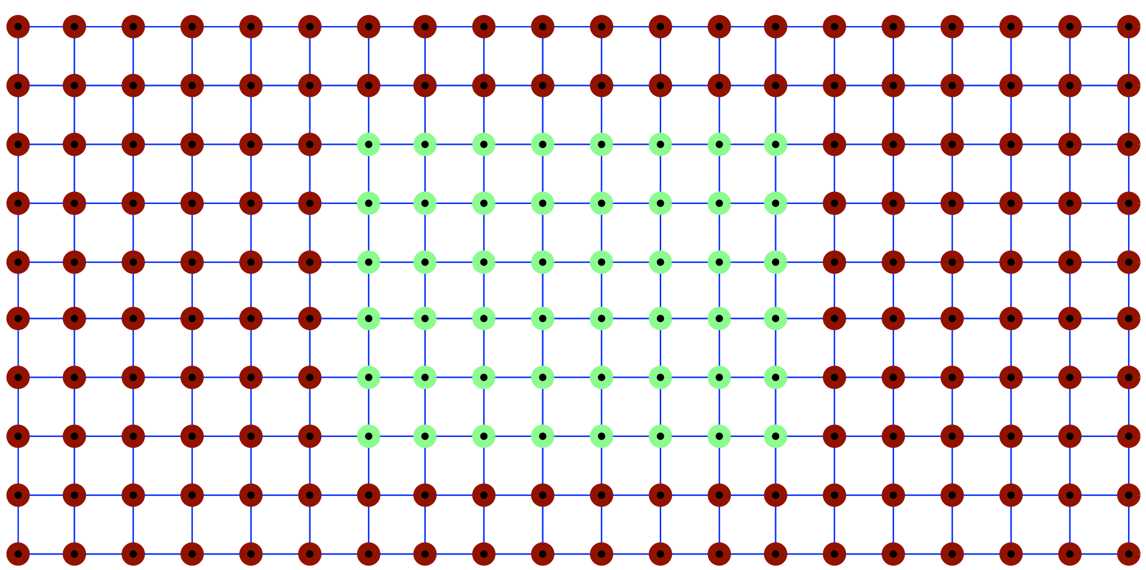
Inside the CS Reduction – Graph Partitioning



Data Graph

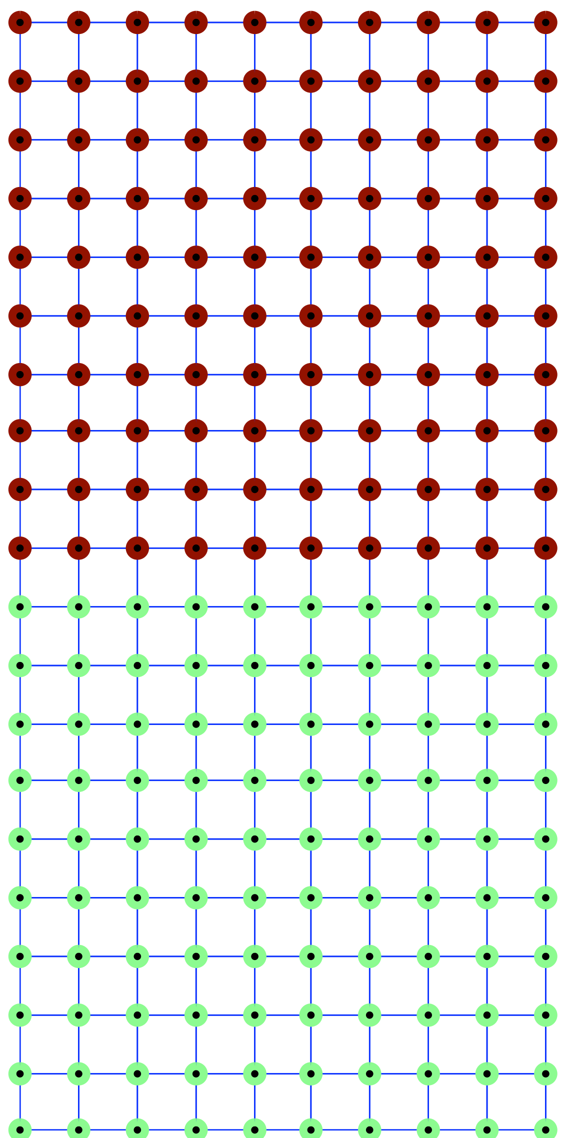


Standard Algorithm

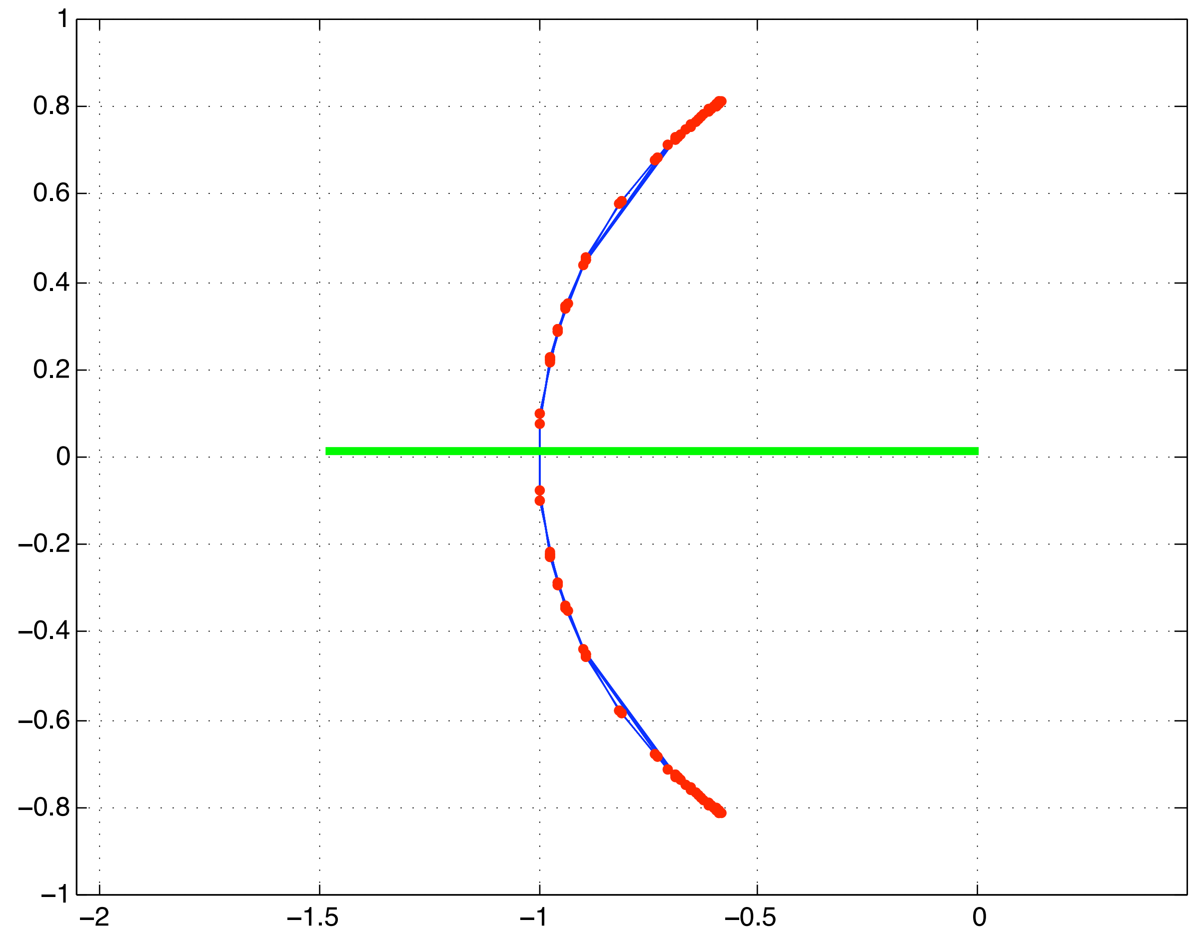


Spectral Rounding

The Standard Spring Model Algorithm

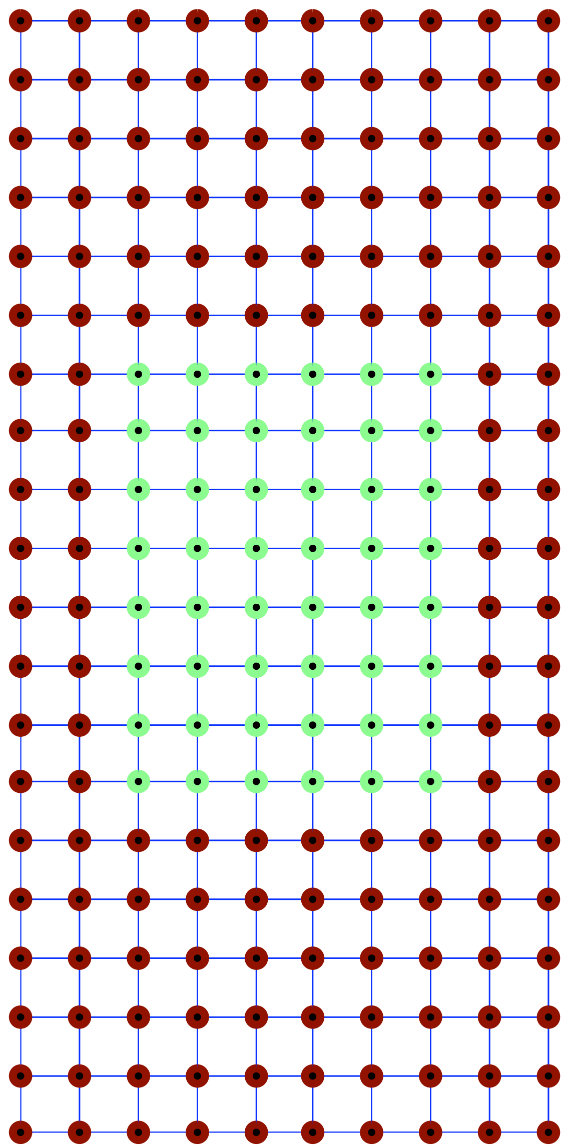


Result - Standard Algorithm

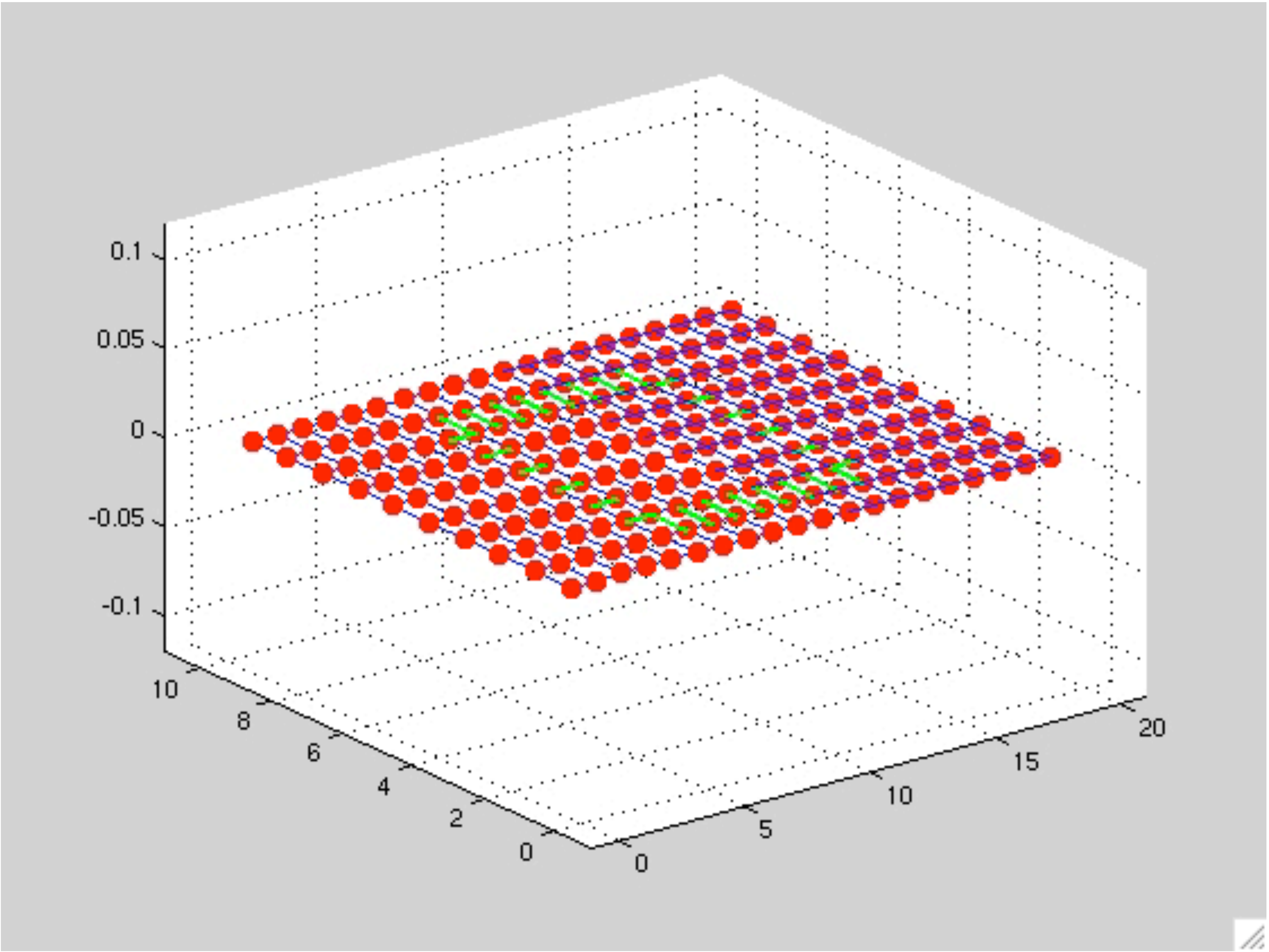


Representation used by the Standard Algorithm

Our Technology



Result - Spectral Rounding



SR leverages the physical intuition!

Mathematical Formulation

- Ohm's Law and Graph Laplacians
- Let $A_{ij} = w_{ij}$ and $D_{ii} = \sum_j w_{ij}$
- The Laplacian

$$L = D - A$$

- Simple fact $LV = I$ where V is voltage I current

Mathematical Formulation

- Solving conductor model problems reduces to solving Laplacians
- Here the Graphs are in fact planar.

Mathematical Formulation for spring models

- We consider the case where the node has mass equal to its weighted degree. The Normalized Laplacian!
- Thus our eigenvalues and vectors satisfy $Lf = \lambda Df$

Mathematical Formulation for spring models

- Zero eigenvalue $L1 = 0D1$
- Rayleigh quotient $$\lambda_2 = \inf_{f \perp D1} \frac{f^T L f}{f^T D f}$$
- Goal: reweight graph to reduce λ_2

Spectral Rounding
Edge reweighting

- Algorithm
  - Solve $Lf = \lambda_2 Df$
  - Reweight graph getting L' and D'
  - Solve $L'f = \lambda_2 D'f$
  - repeat while Lambda not zero
  - repeat while best threshold cut is changing

Spectral Rounding
Finding a good reweighting

Lemma 1. *Given a weighted symmetric graph $G = (V, E, w)$ then the normalized Rayleigh quotient can be written as*

$$\frac{f^T L f}{f^T D f} = \frac{\sum_{(i,j)\in E, i<j} (f_i - f_j)^2 w_{ij}}{\sum_{(i,j)\in E, i<j} ((f_i)^2 + (f_j)^2) w_{ij}} \tag{3.3}$$

where $f_i = f(v_i)$

Finding a good reweighting
Mediant of fractions

Definition 1. *Given formal fractions*

$$\frac{a_1}{b_1}, \cdots, \frac{a_n}{b_n}$$

the **fractional average** *is the formal fraction*

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

where the a_i's and b_i's are reals.

Finding a good reweighting using Mediant of fractions

Lemma 2. *If* $\frac{a_1}{b_1} \leq \cdots \leq \frac{a_n}{b_n}$ *and* $w_1 \geq \cdots \geq w_n$ *then*

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \geq \frac{\sum_{i=1}^n a_i w_i}{\sum_{i=1}^n b_i w_i}$$

The inequality is strict if for some pair $1 \geq i < j \leq n$ *we have that* $\frac{a_i}{b_i} < \frac{a_j}{b_j}$ *and* $w_i > w_j$.

Inverse Fractional Reweighting

- Given $\mathrm{L \text{ and } D}$ we get $\mathrm{L' \text{ and } D'}$
- where $$w'_{ij} = \frac{f_i^2 + f_j^2}{(f_i - f_j)^2} w_{ij}$$
- Gives $$\frac{f^T L f}{f^T D f} \geq \frac{f^T L' f}{f^T D' f}$$
- Problem: in general $$\lambda_2 \not\geq \lambda'_2$$

1D Family of Matrices

- 1D family $W(t) = W + tW'$

- Theorem:

$$\lambda = \frac{f^T L f}{f D f} > \frac{f^T L' f}{f^T D' f} \quad \text{implies} \quad \frac{d\lambda(t)}{dt} < 0$$

Defining Segmentation Quality

- Two Measures of Quality
  - A Mathematical Quantity e.g. Normalized Cut(NC)
  - Human hand segmentation

We do well with respect to both measures

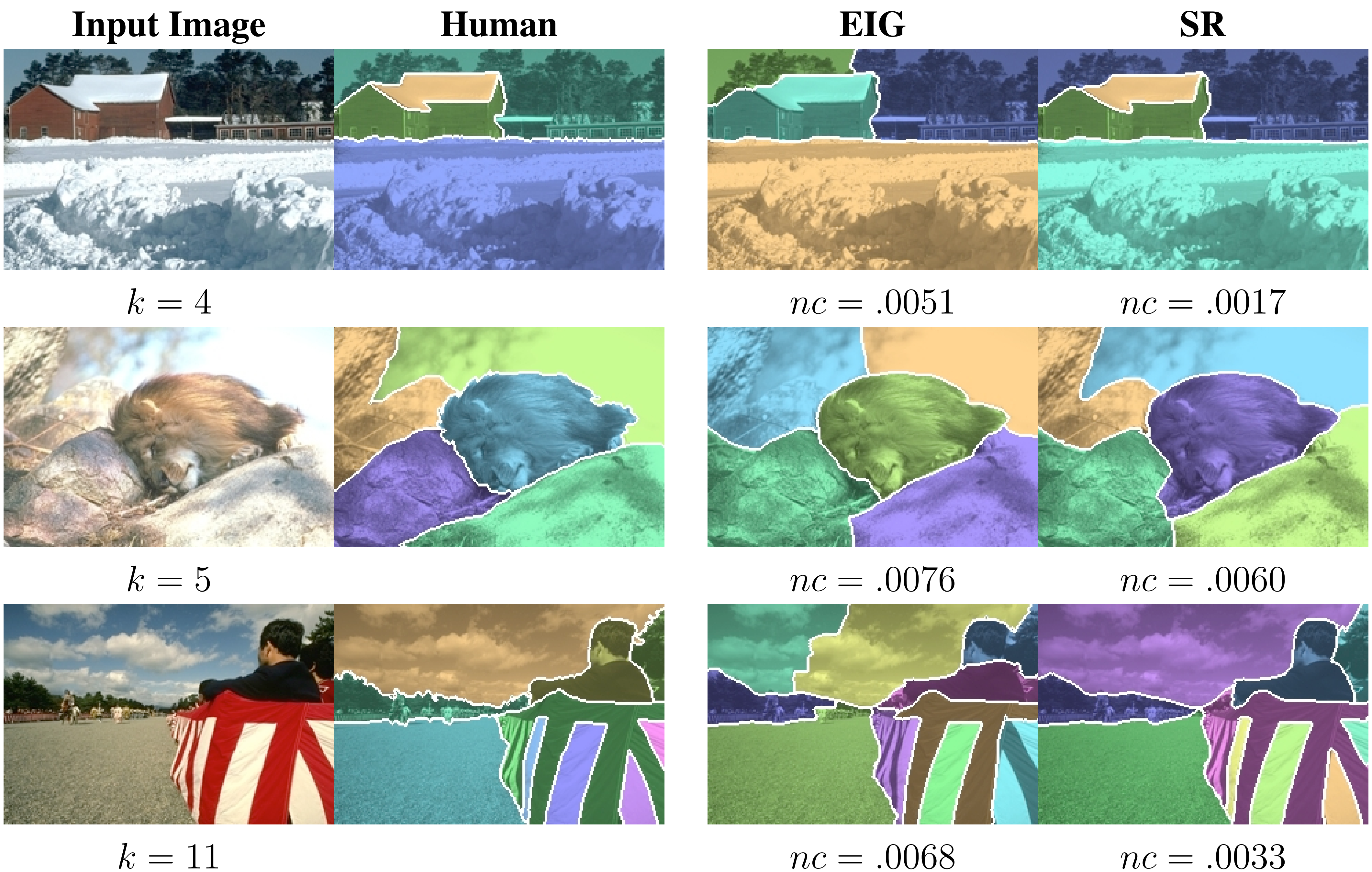
Normalized Cut

- Definition:

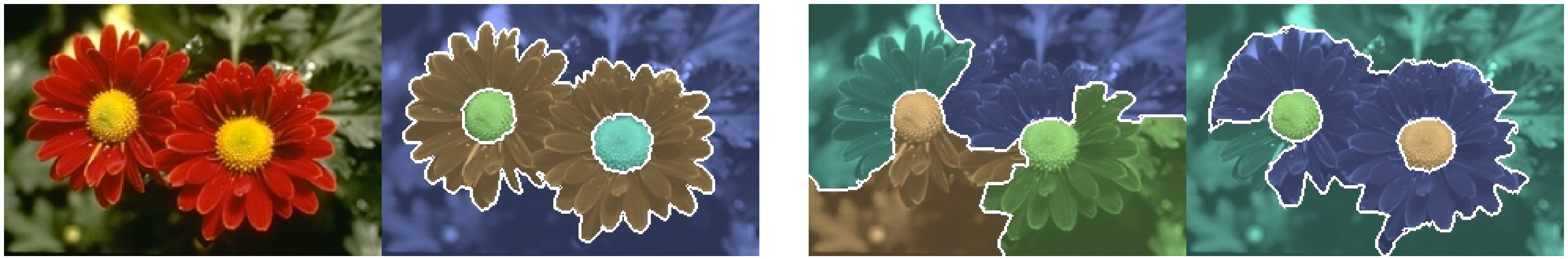
$$nc(G) = \min_{V_1,..,V_k} \frac{1}{k} \sum_{i=1}^{k} \frac{cut(V_i, V \setminus V_i)}{vol(V_i)}$$

Where $V_1 \cdots V_k$ is a Partition of V.

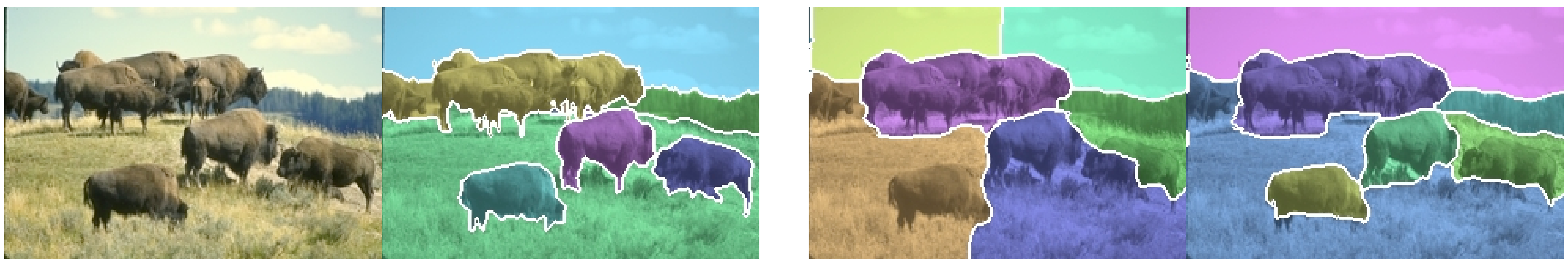
Comparison with Human Segmentation



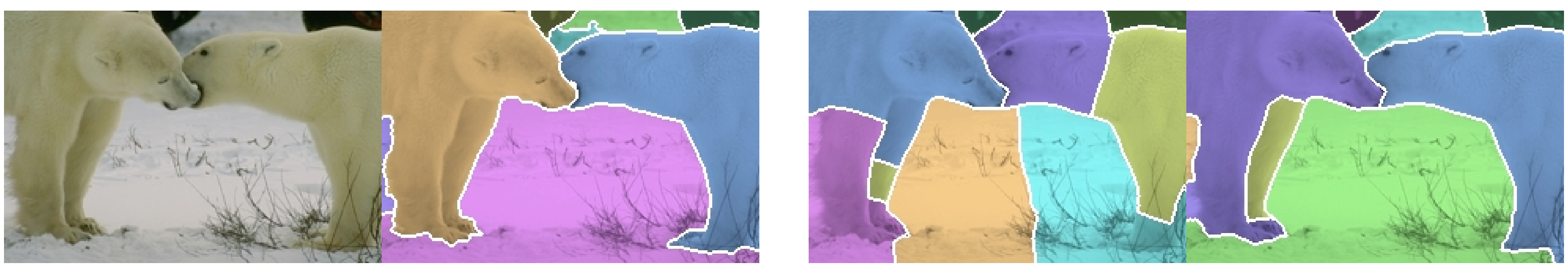
Comparison with Human Segmentation



$k = 4$ $nc = .0039$ $nc = .0009$

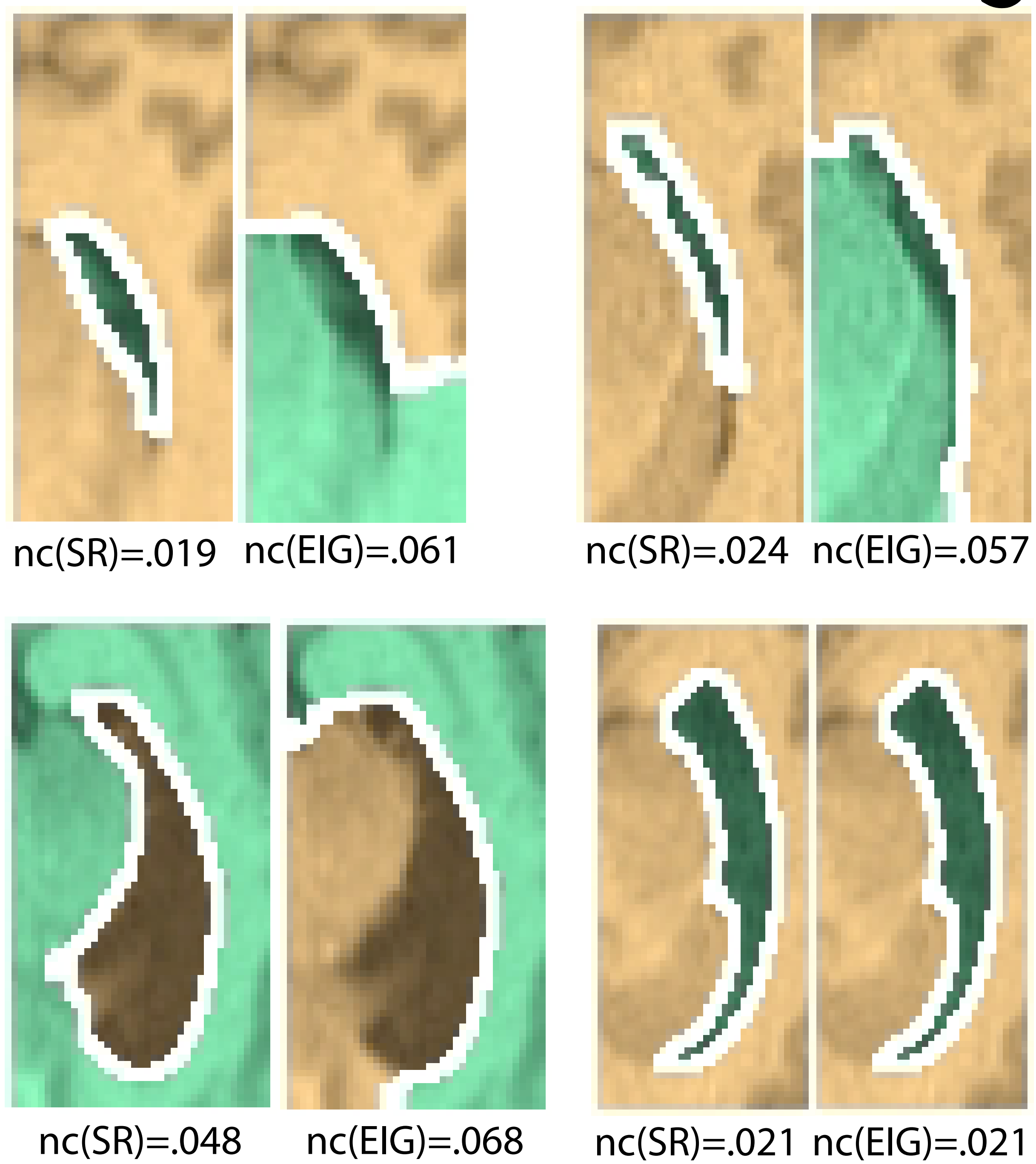


$k = 7$ $nc = .0023$ $nc = .0011$



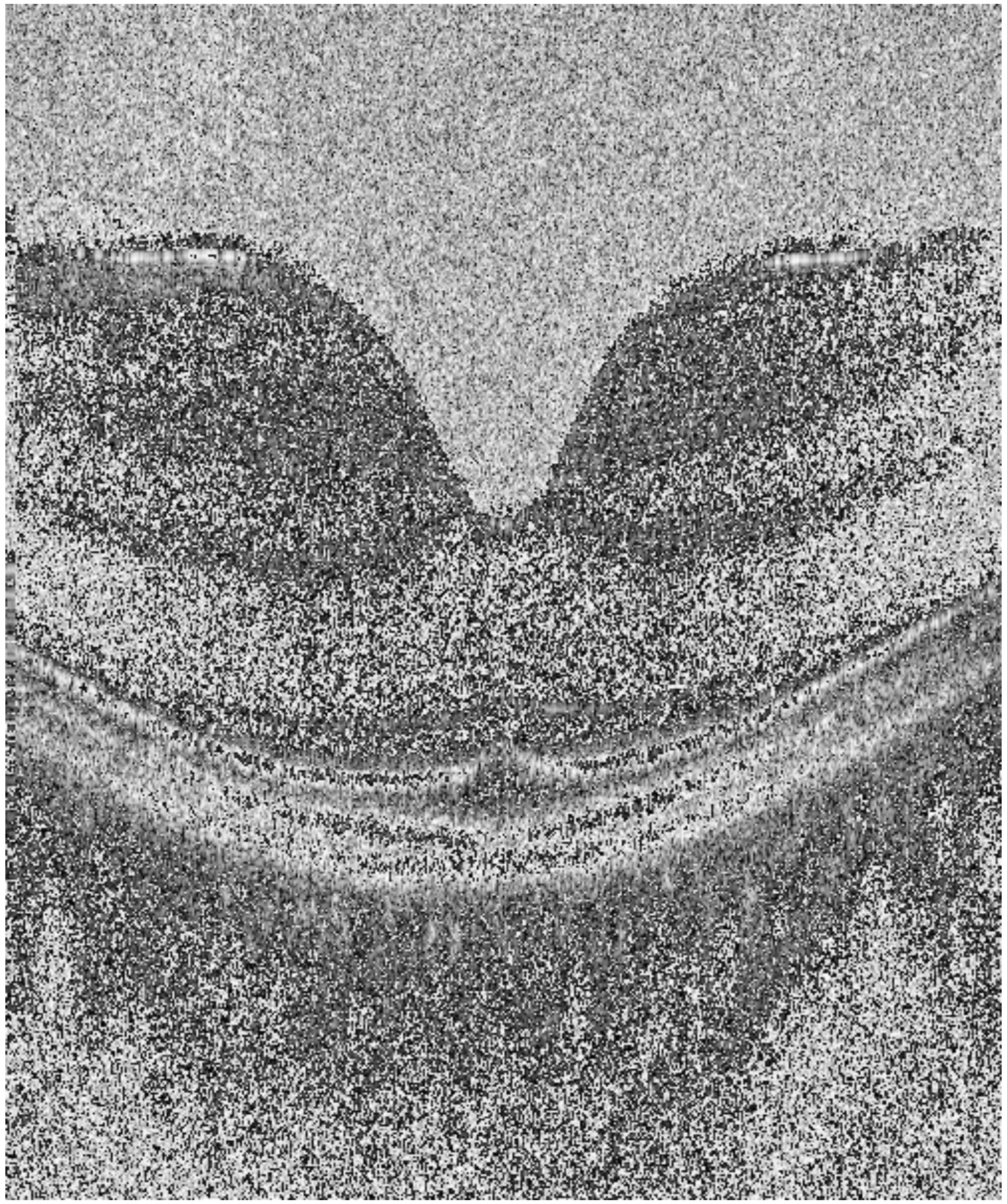
$k = 8$ $nc = .0012$ $nc = .010$

Results: Medical Images

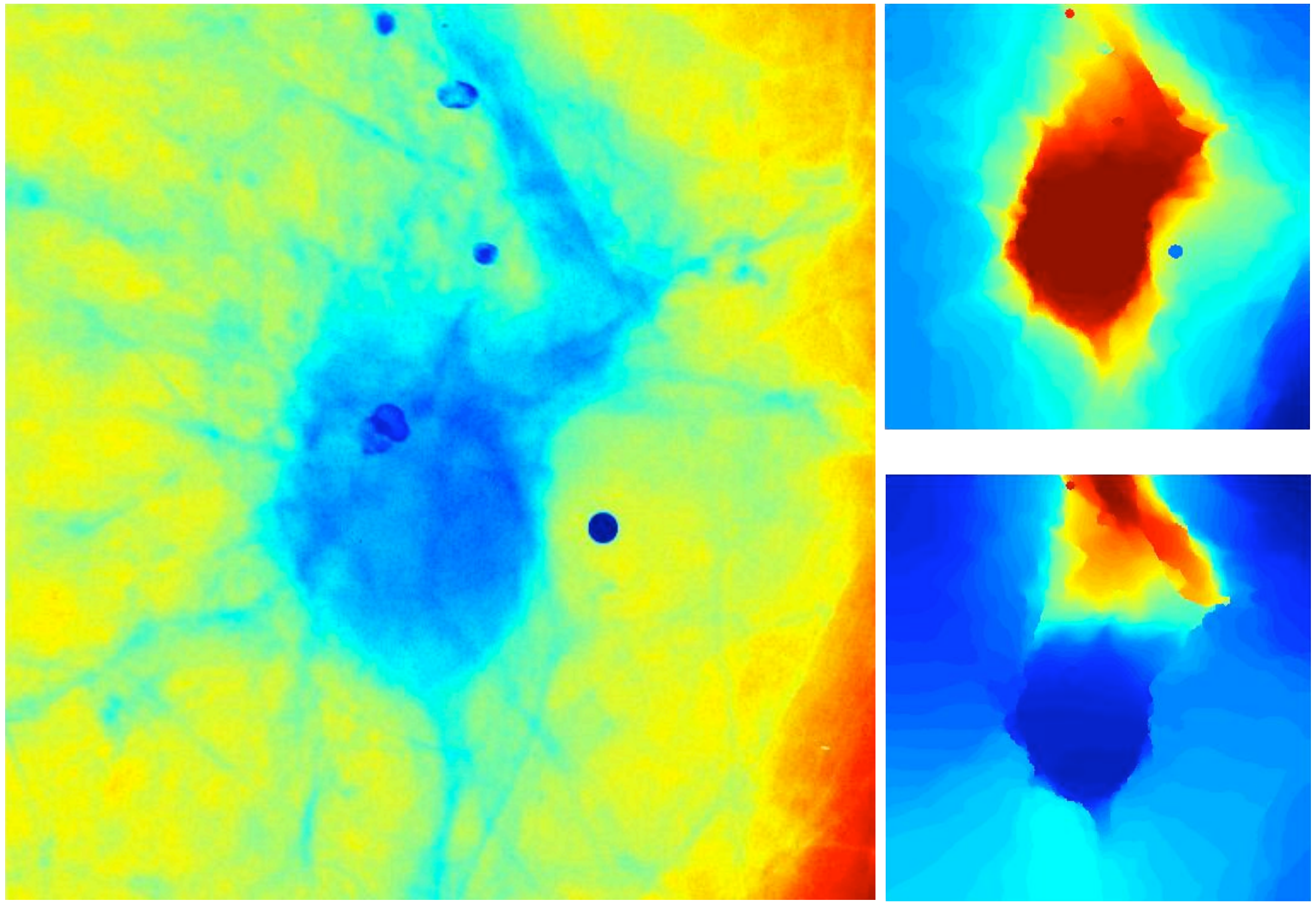


MRI data of left ventricle

Medical Segmentation



retinal volume processing



assisted tumor extraction

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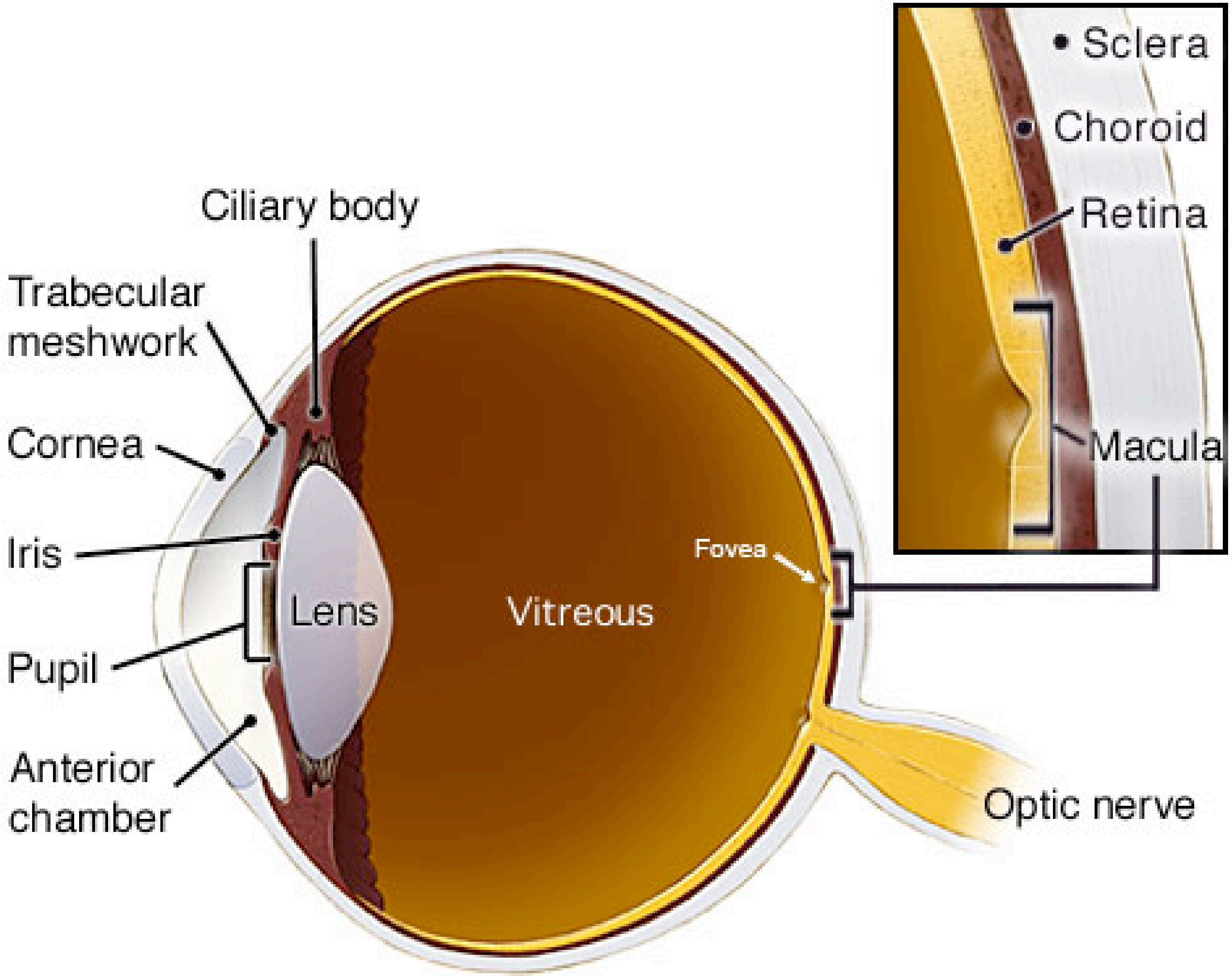
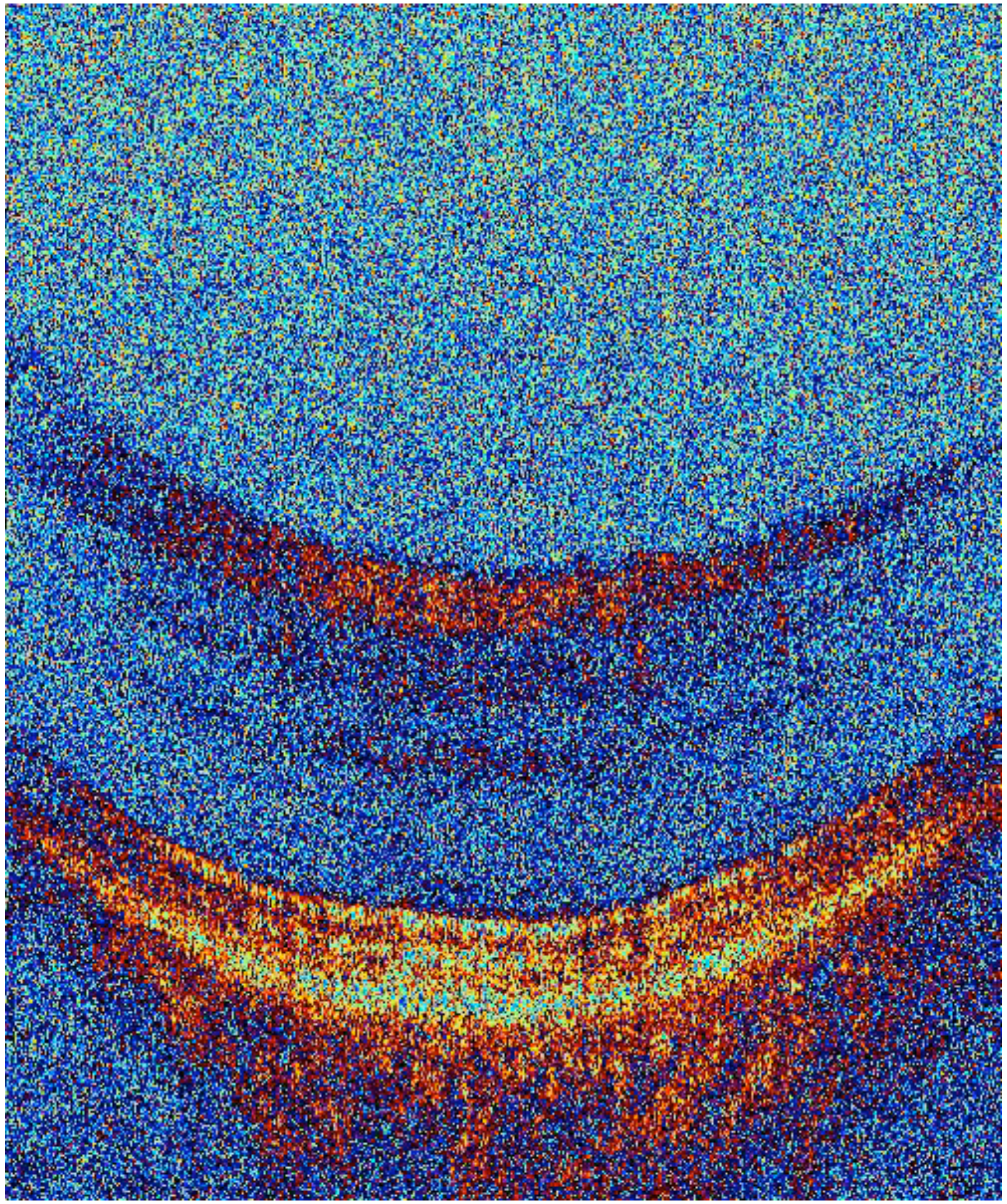
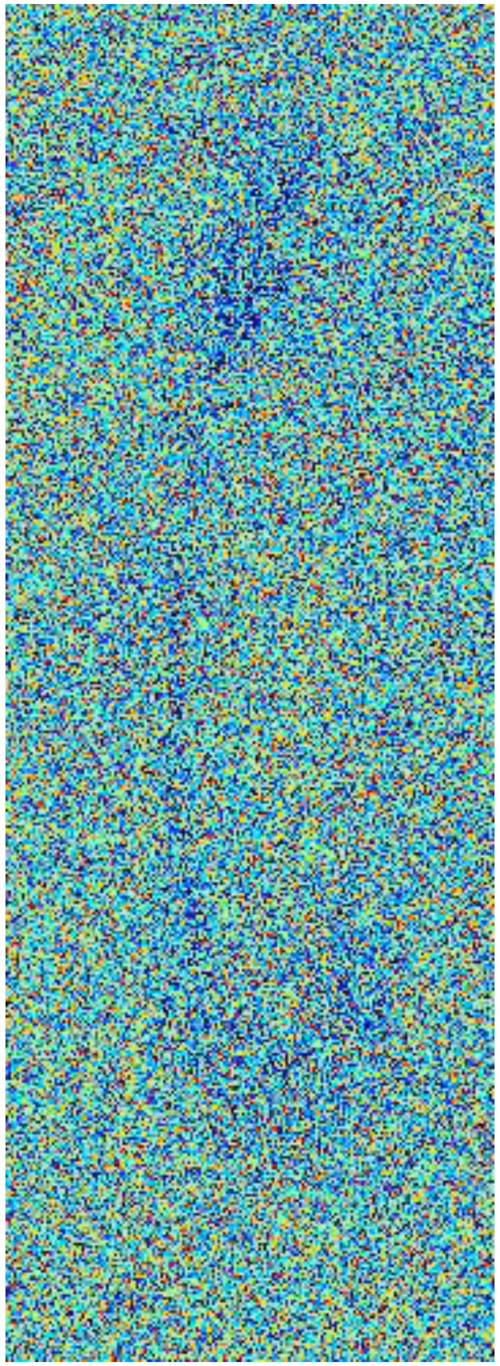
Anatomy of the Eye



Spectral OCT

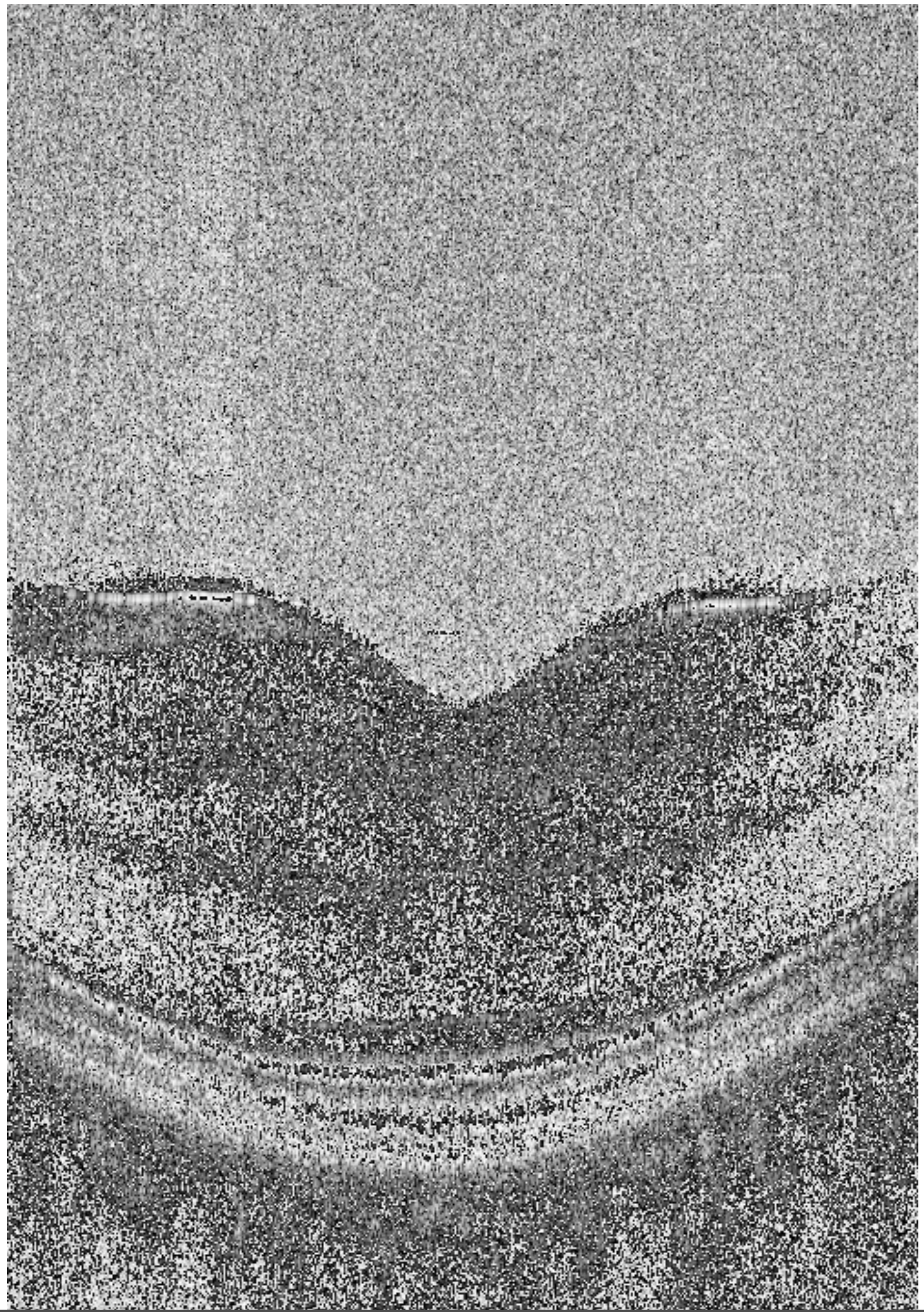


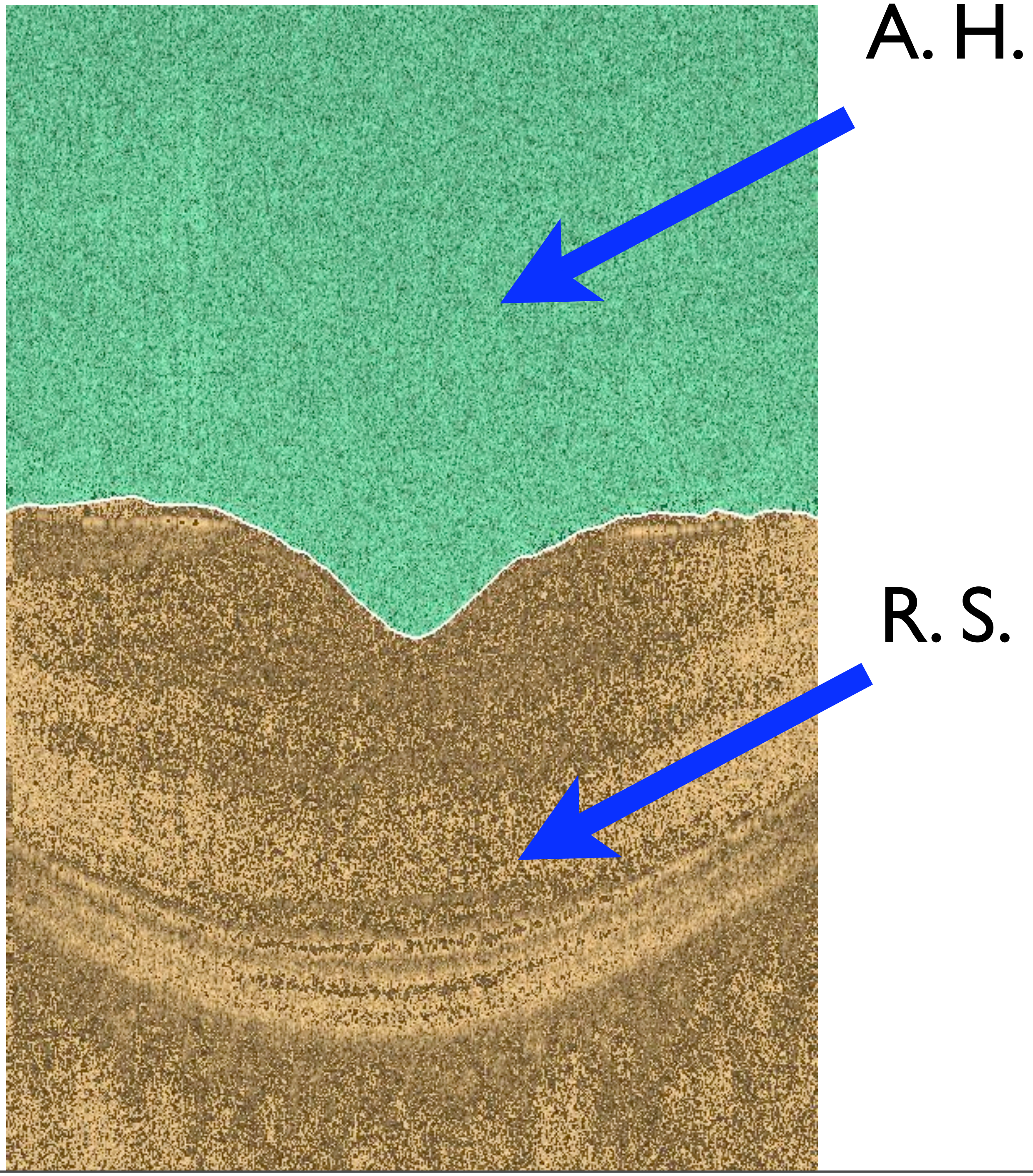


Spectral Rounding: *Global* vs. ***Local***

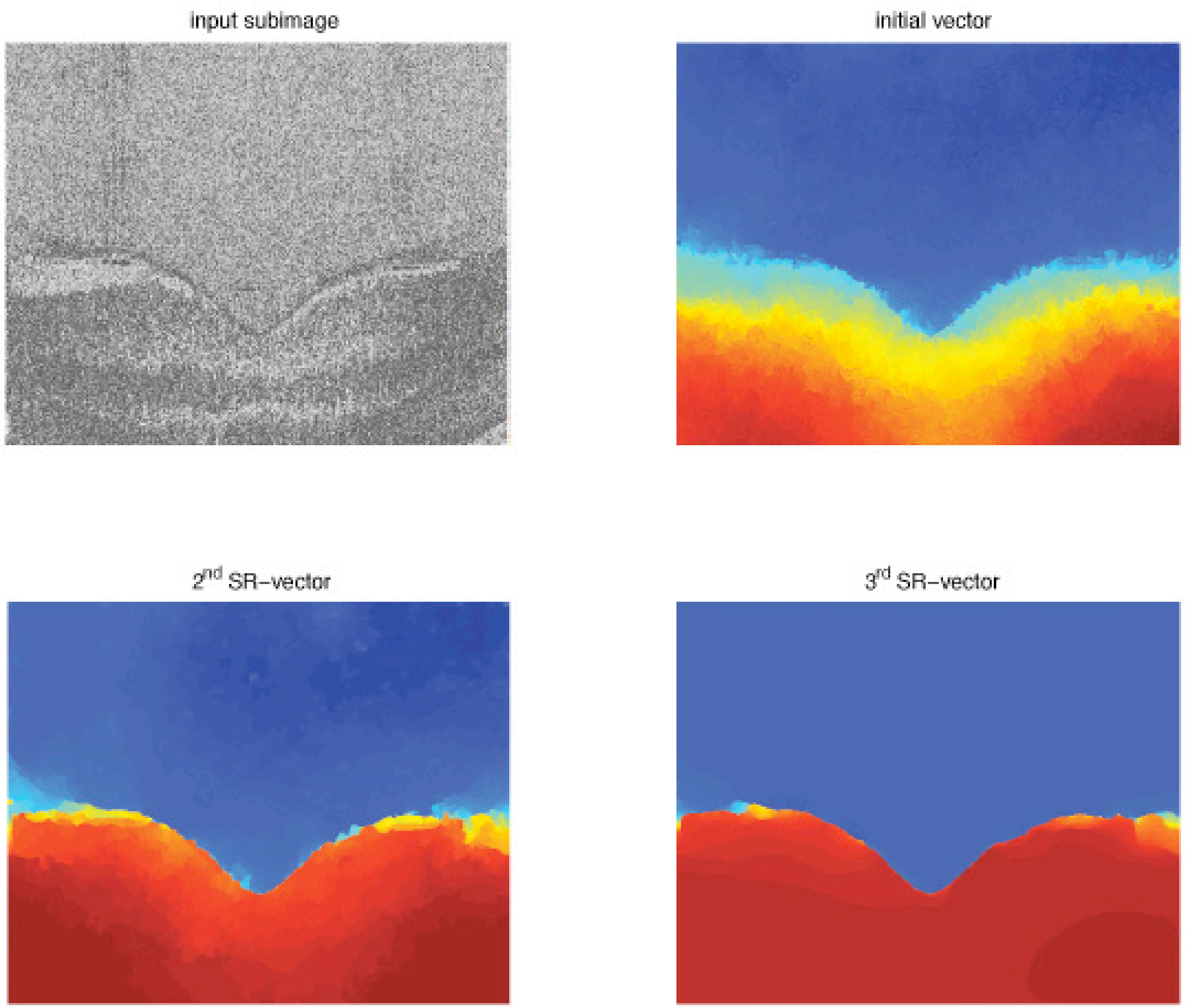
Threshold - common in MIP

Spectral Rounding: ***Global*** *vs. Local*

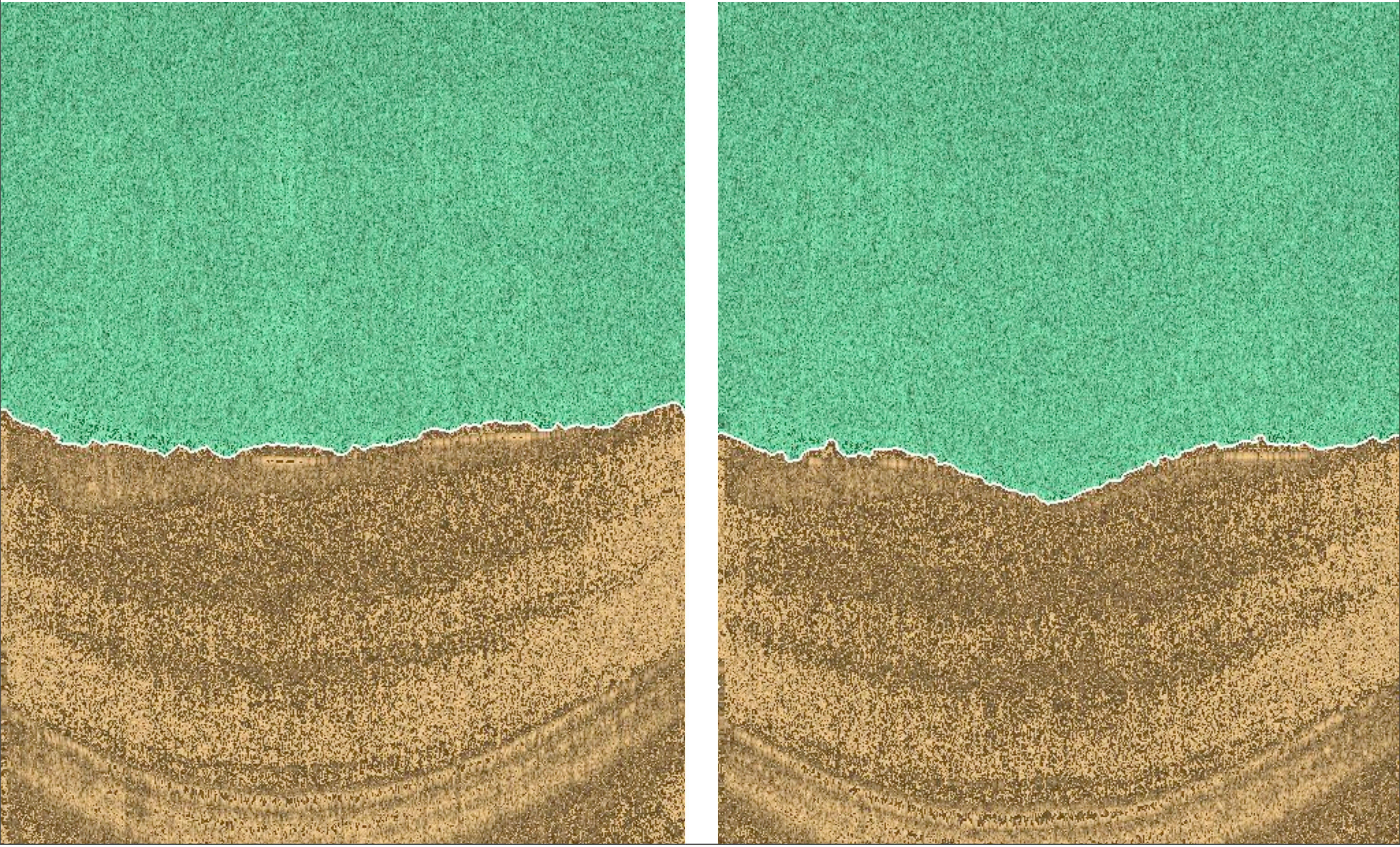




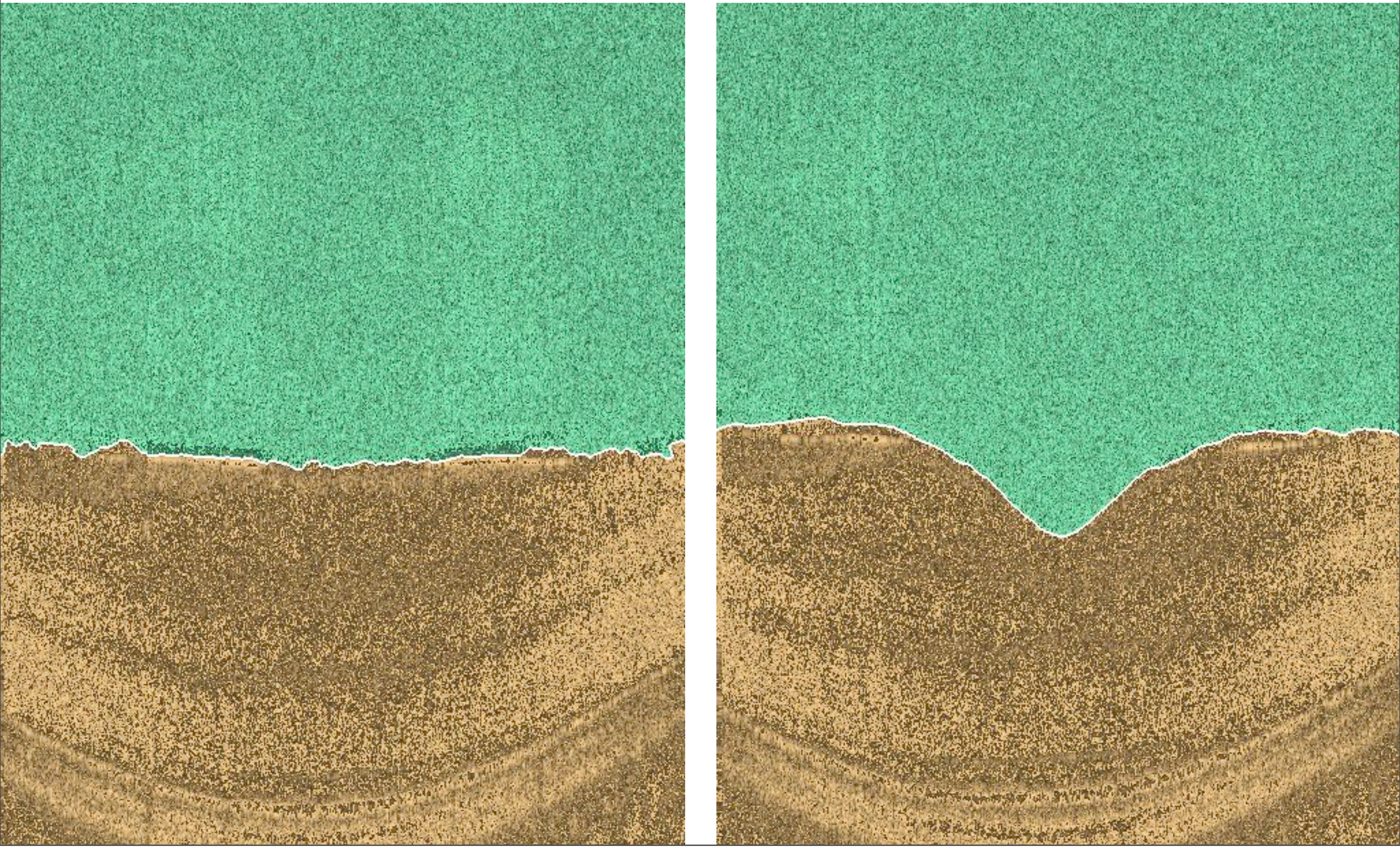
S.R. in action...



Segmentation Results



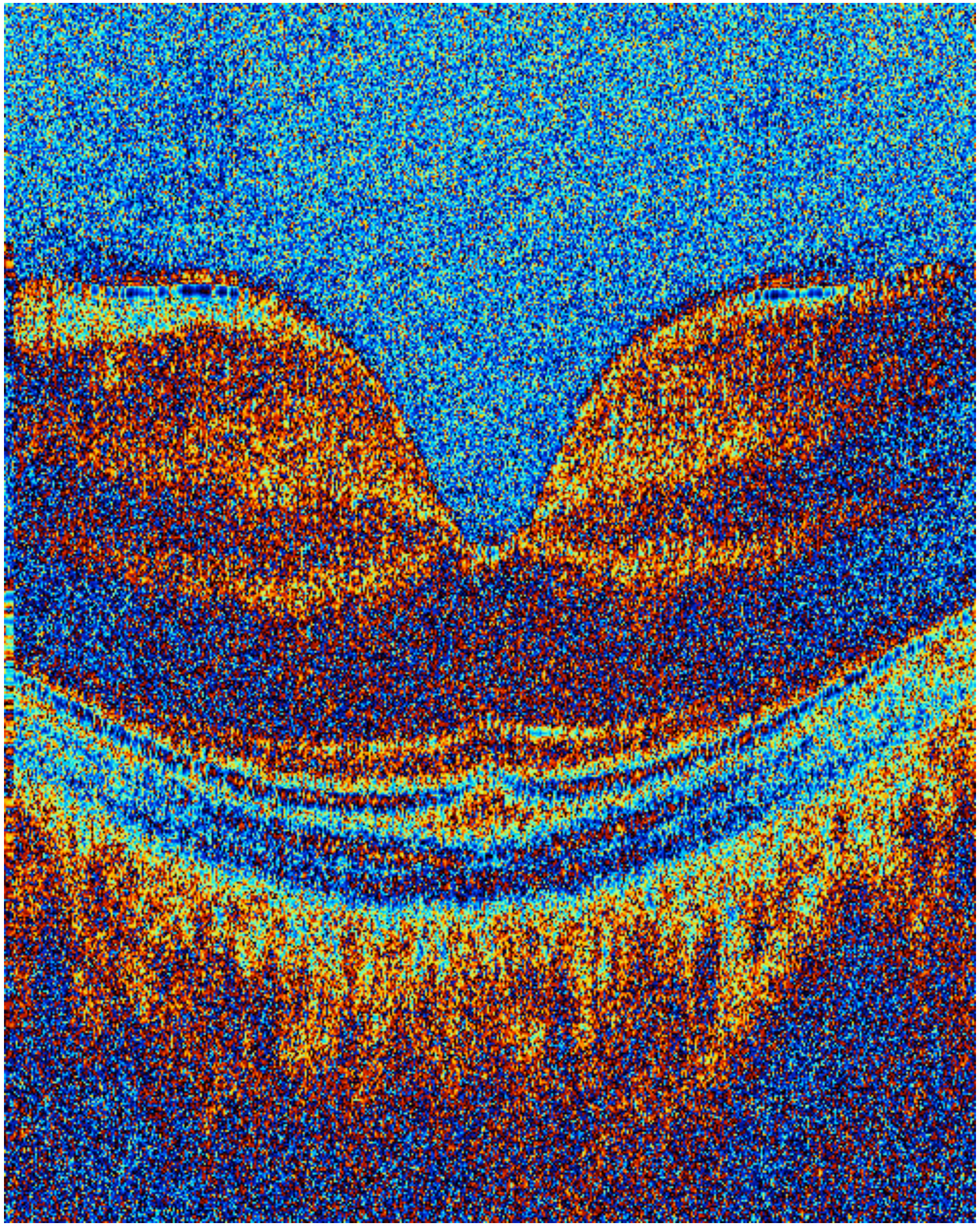
Segmentation Results



Fly Through

NFL Extraction:

Detection of the Nerve Fiber Layer Contour



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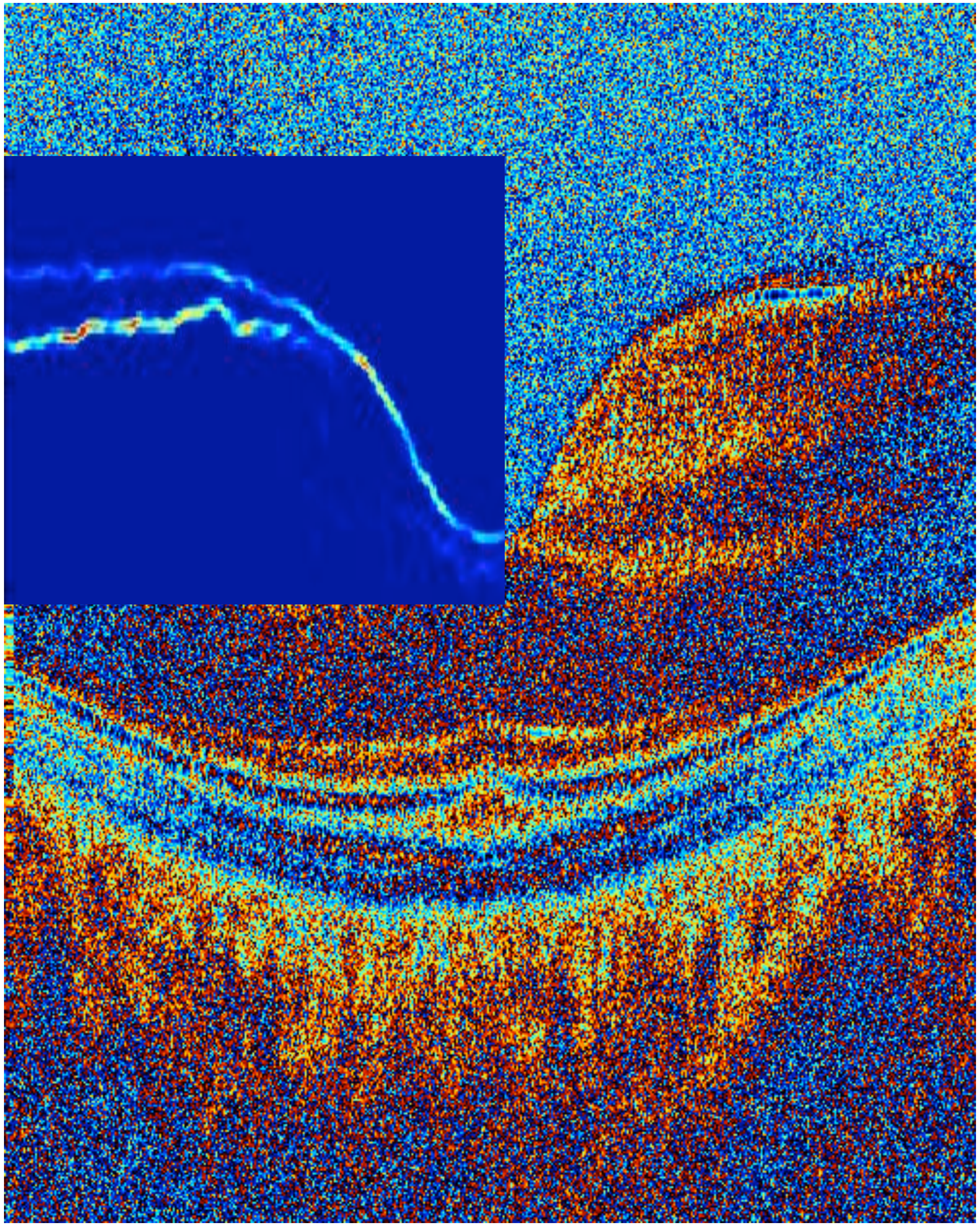
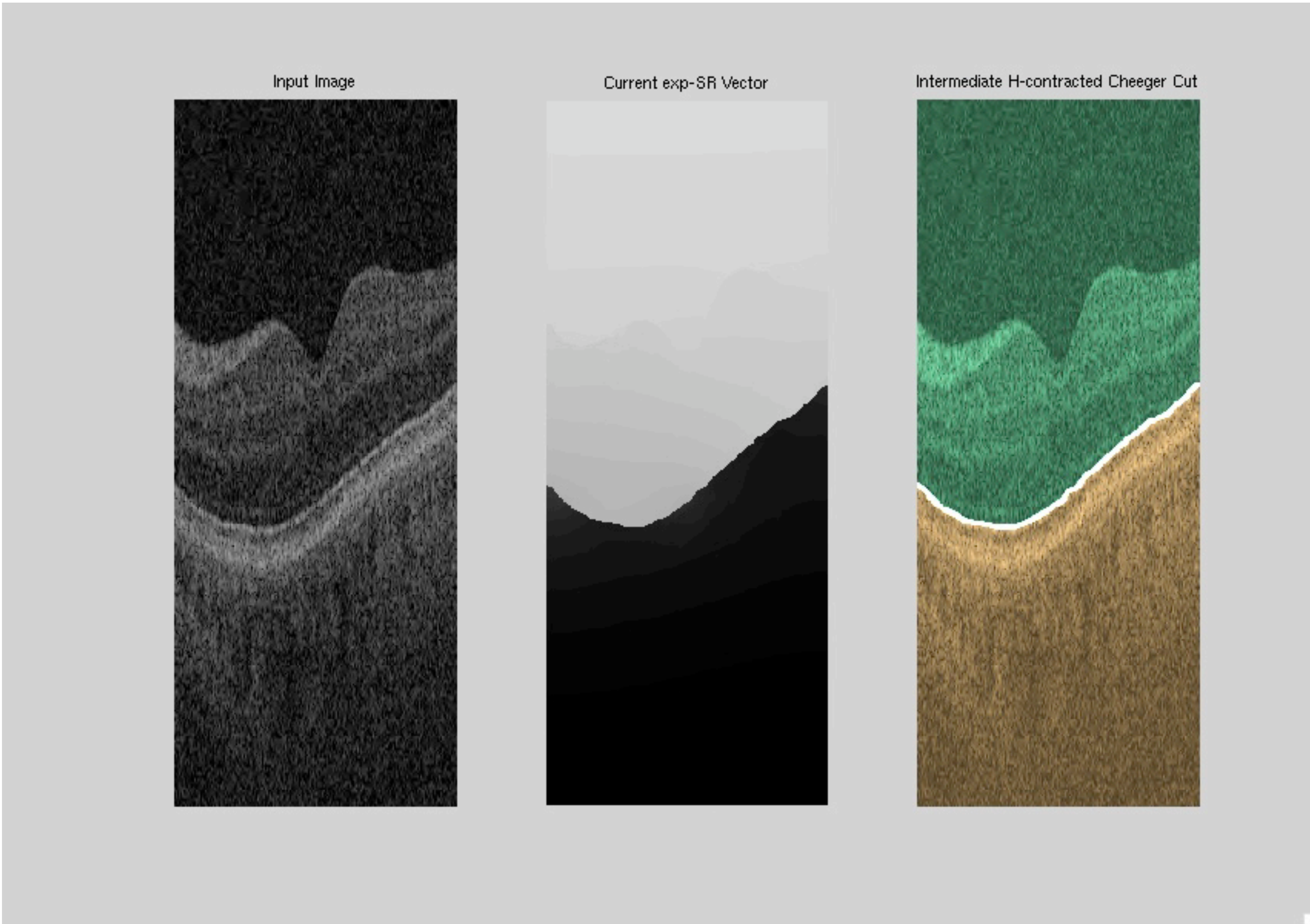
NFL Extraction:

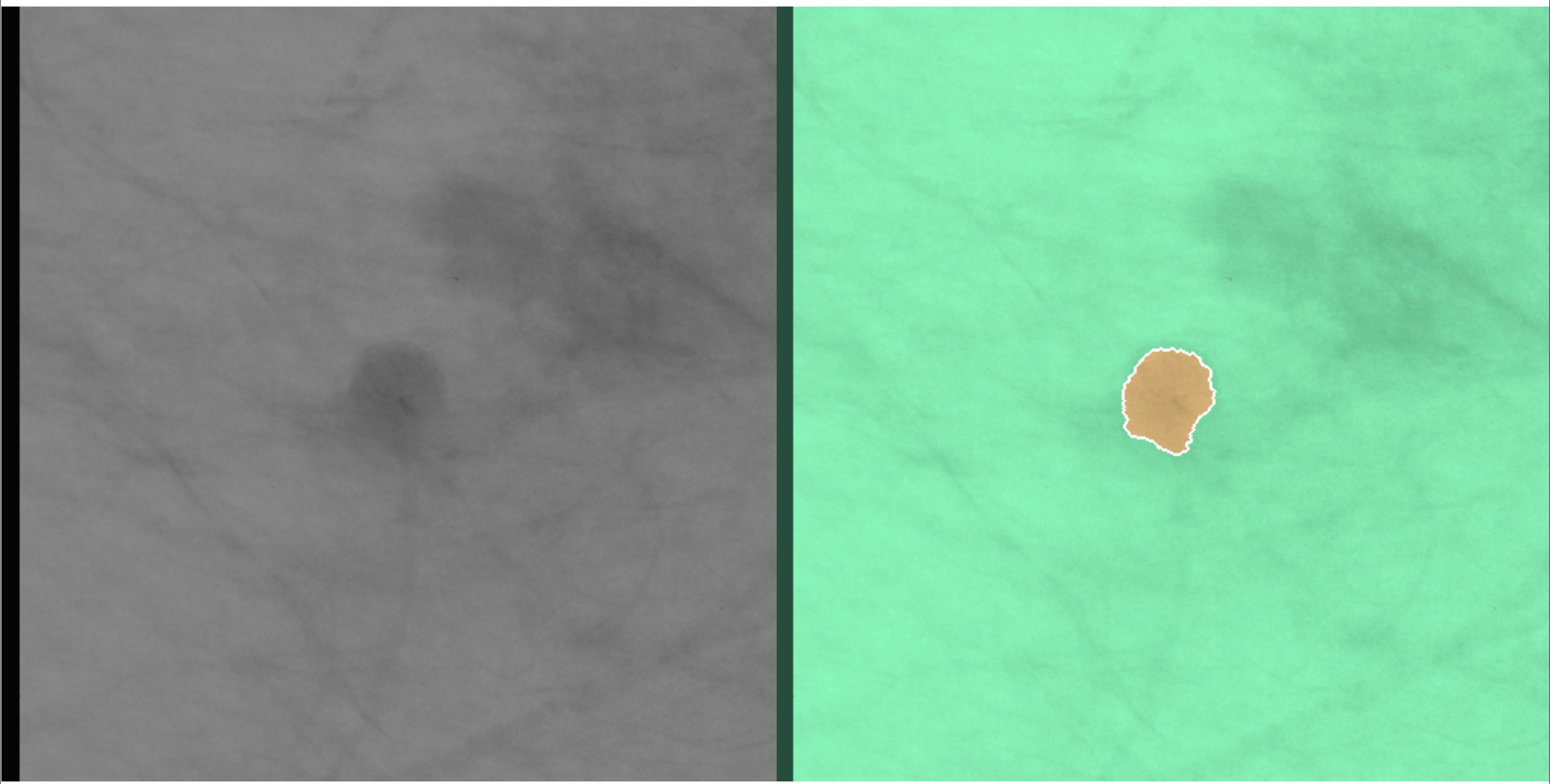
Intensity proportional to *probability-of-a-cut* under the eigen-space

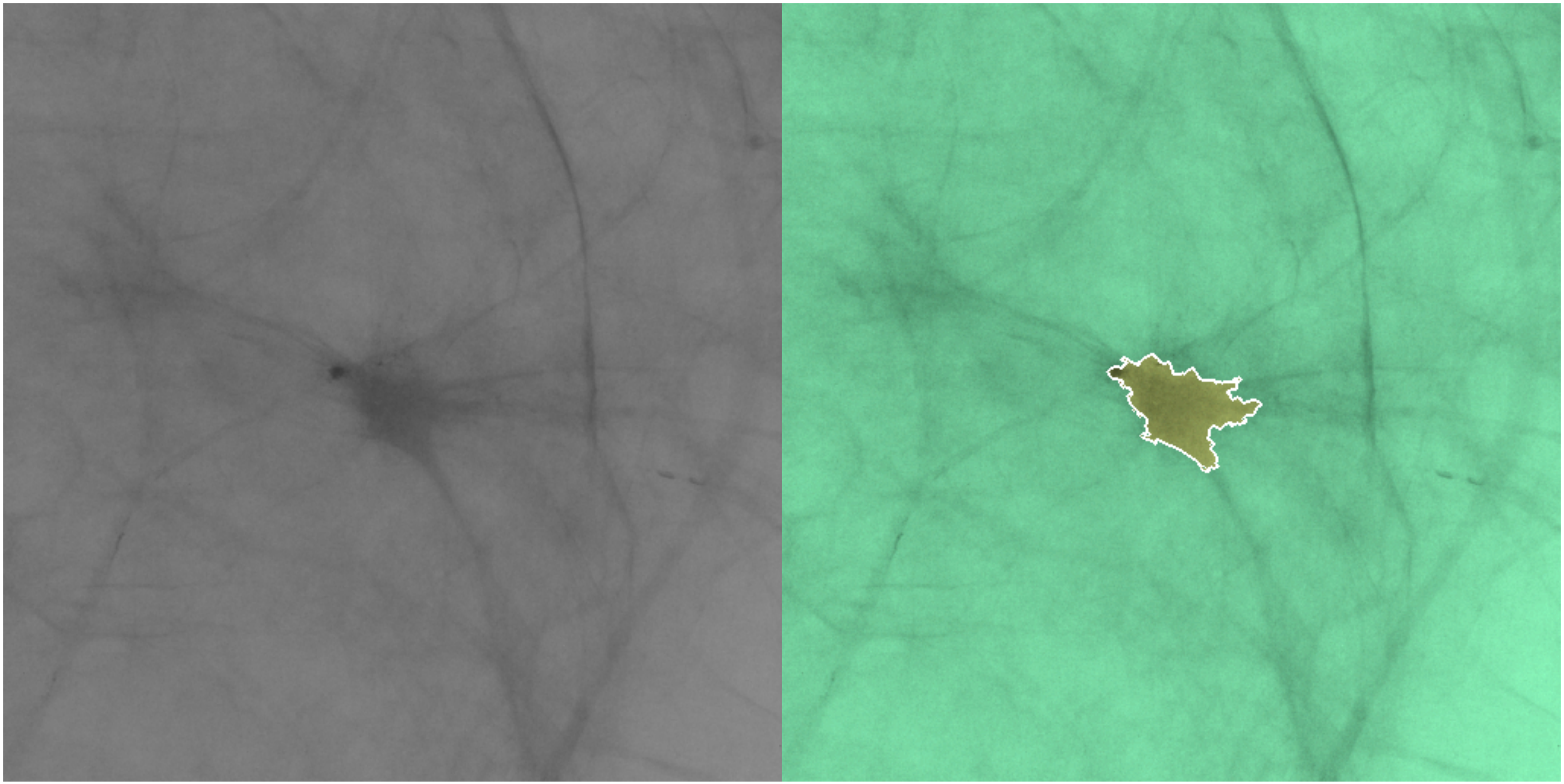


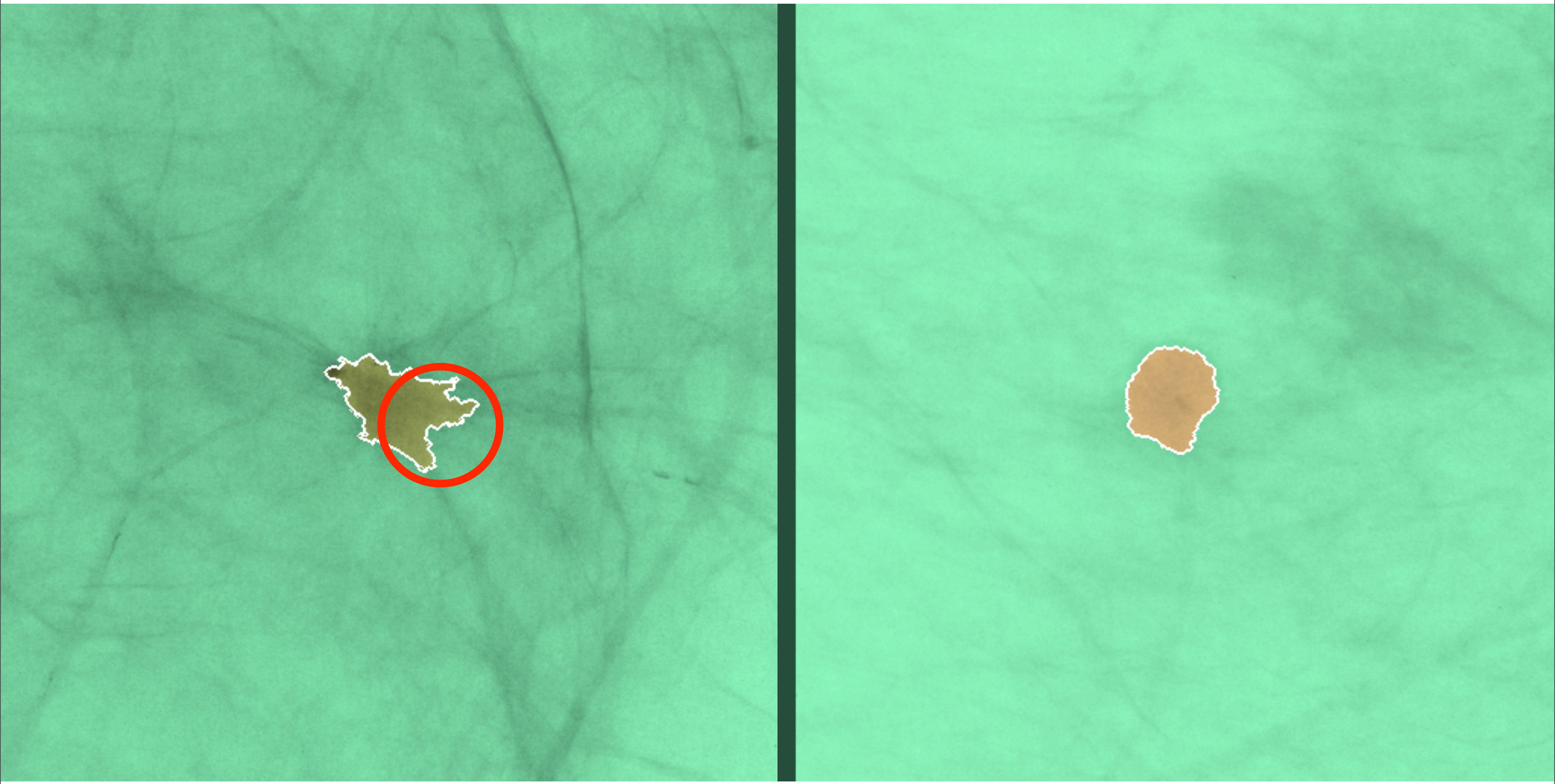
Another view of SR



Mammogram Segmentations







Numerical Algorithms

- Solving Laplacian Lx = b
- Finding eigenvectors $Lf = \lambda_2 Df$
- Spielman and Teng
  - O(n $\log^k$ n) time for some k.

Iterative Solvers $Ax = b$

- Richardson: $x^{(i+1)} \leftarrow (I - A)x^{(i)} + b$
- Preconditioned: $B^{-1}Ax = B^{-1}b = b'$
  $x^{(i+1)} \leftarrow (I - B^{-1}A)x^{(i)} + b'$
- Computing $z = B^{-1}Ax^{(i)}$
  - $y \leftarrow Ax^{(i)}$
  - solve $Bz = y$

Combinatorial Preconditioners

- Recall: A graph G, B graph H
- Vaidya: Max Weight Spanning Tree.
- EEST: Low Stretch Spanning Tree.
- Gremban-M: Steiner Tree

All these generate one for all of G

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Planar Solvers

1. The speed of planar solvers has been dramatically improving over the last 50 years.
2. We have an optimal sequential time algorithm.
3. It also can be used on in parallel.

Dealing with larger images

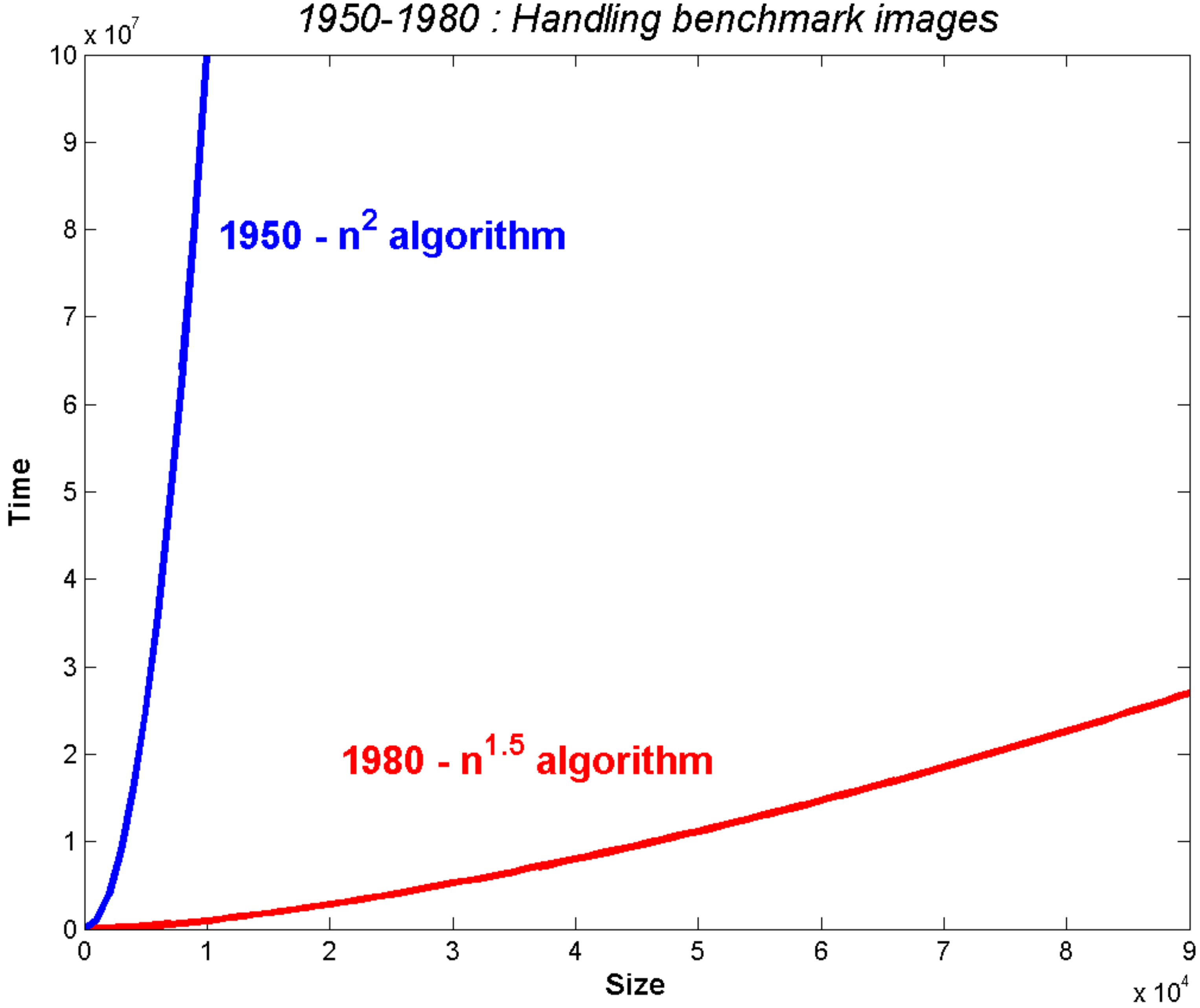
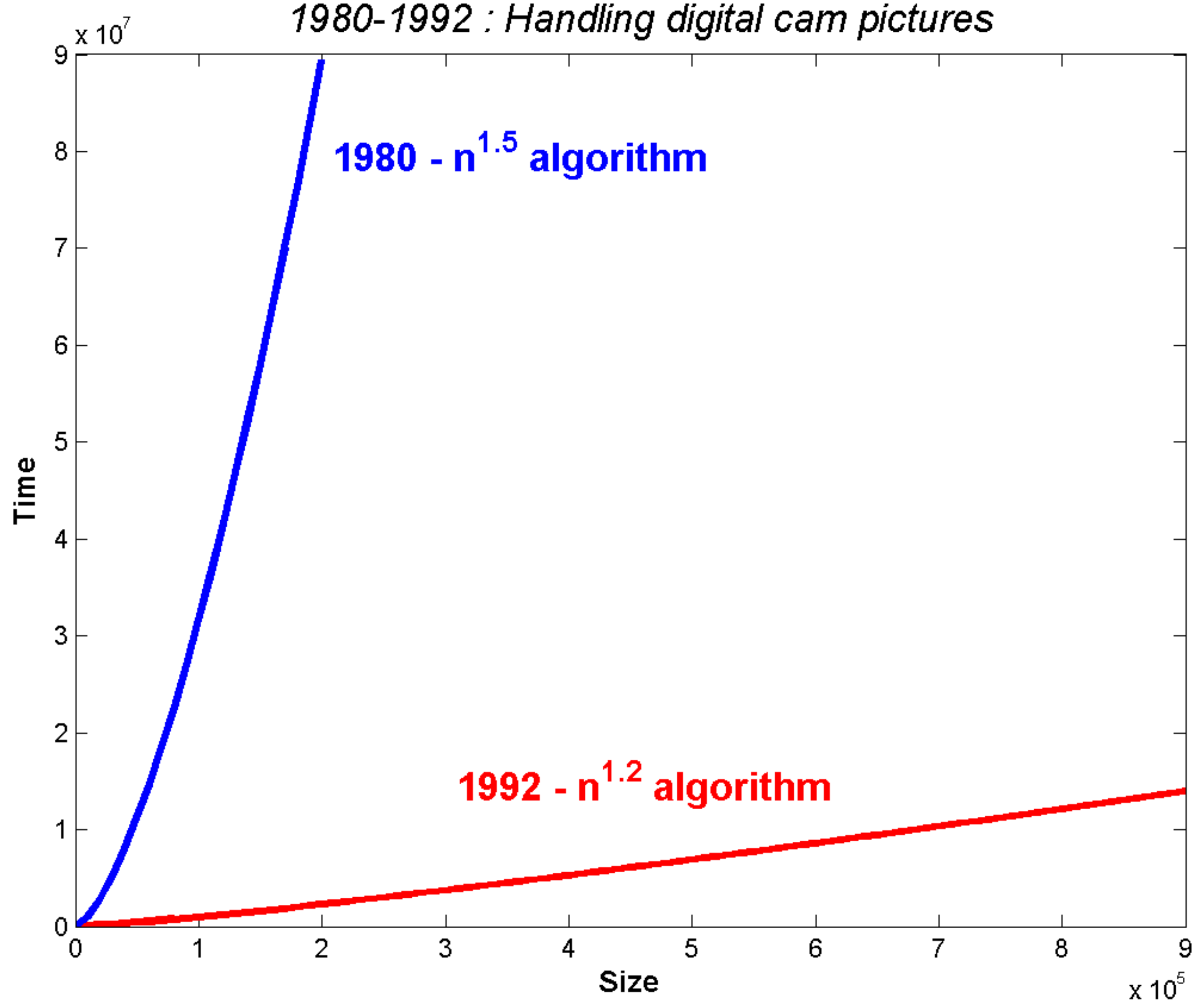


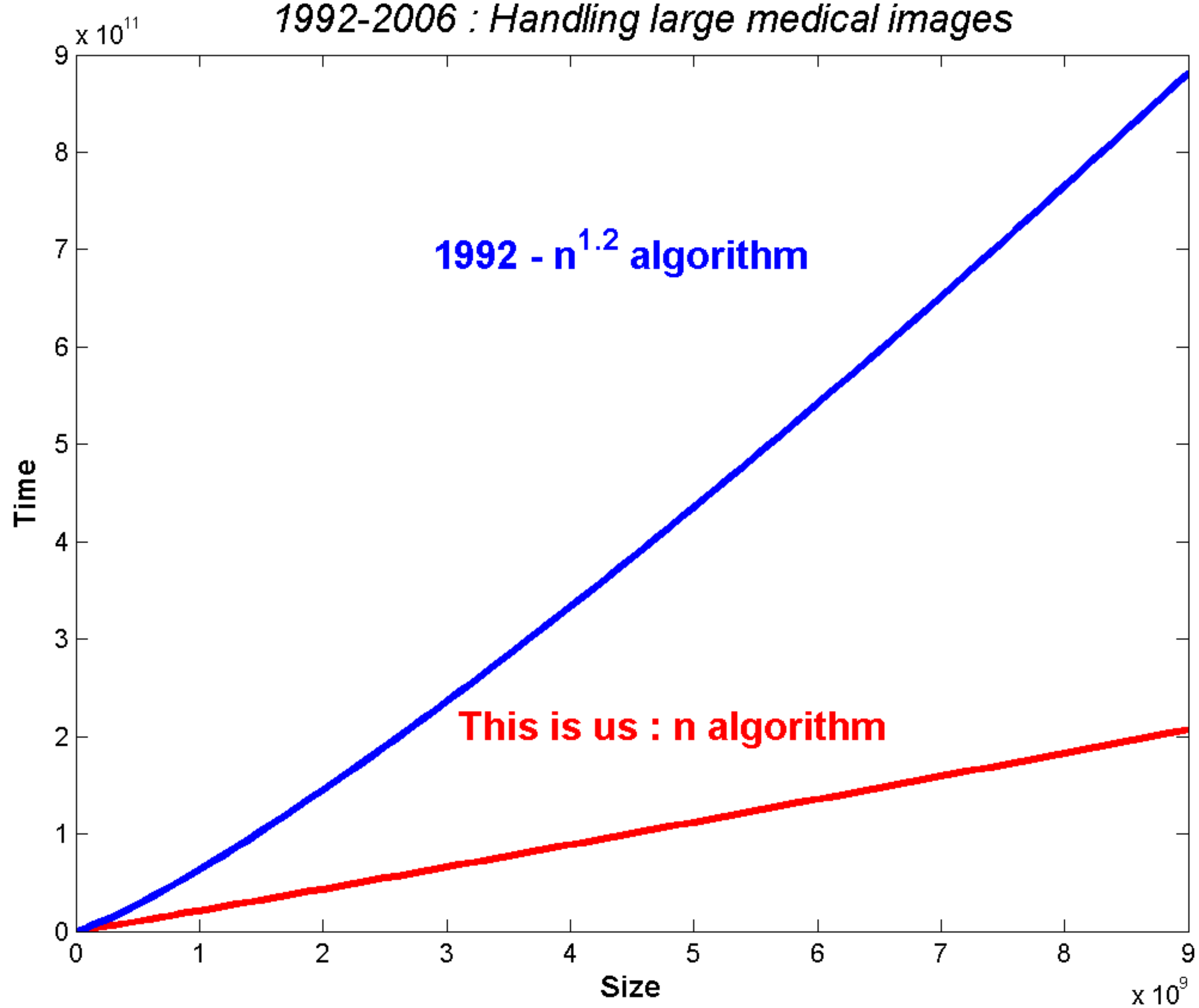
Image sizes up to 100K pixels

Dealing with larger images



Images up to 1 mega-pixel

Dealing with larger images



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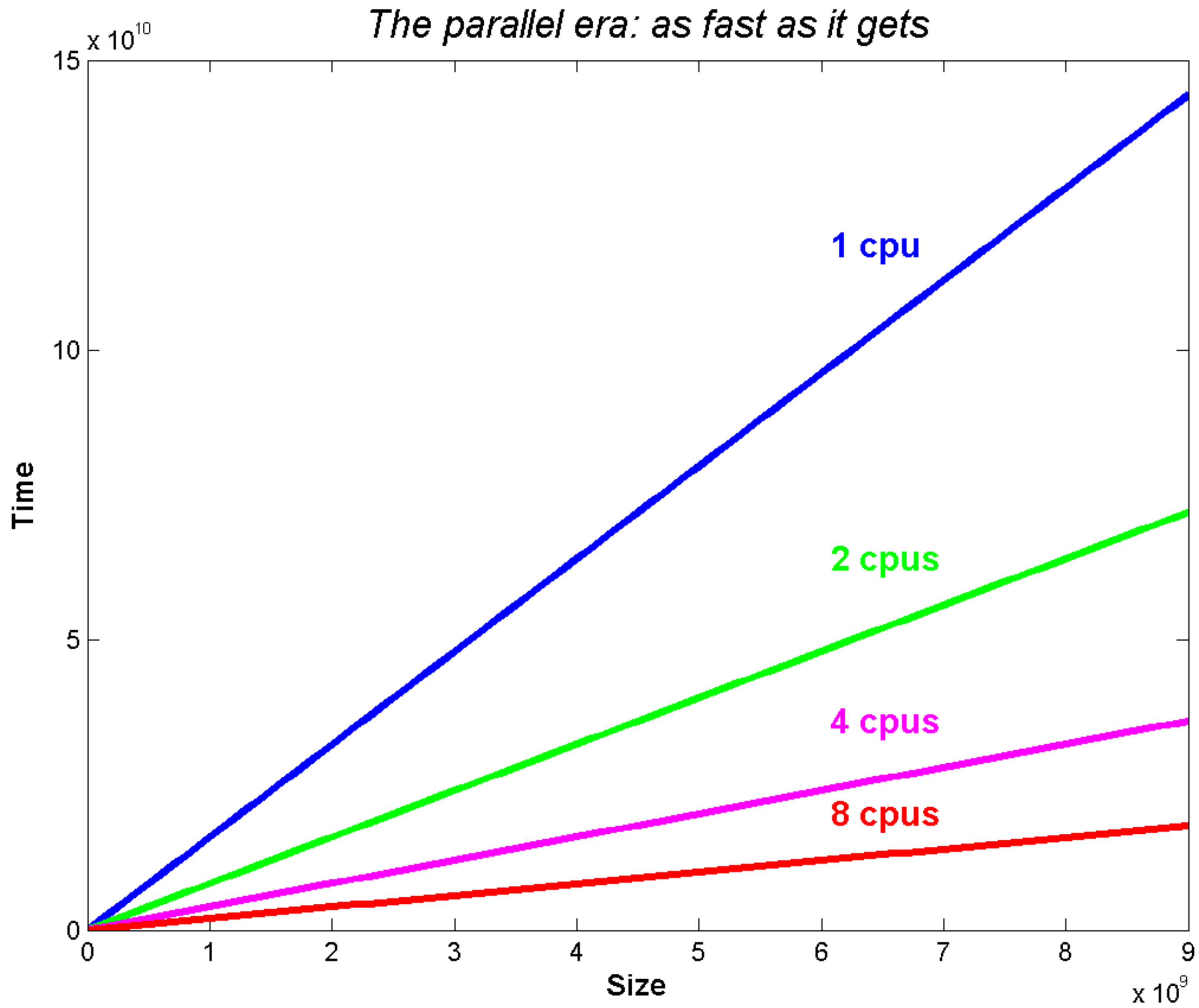

Giga-pixel images and beyond...

Our Preconditioner

- Partition G into small pieces with small boundary.
- Use one of the known preconditioners for each piece.

Our Partitioner

Planar $G = (V,E)$

- Partition $P_{1, \ldots.} P_m$ of E
- $|P_i| <= k$
- sum over bdaries <= O(n/ sqrt k)
- Work: O(n)
- Time: O(k log n)

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Dealing with larger images

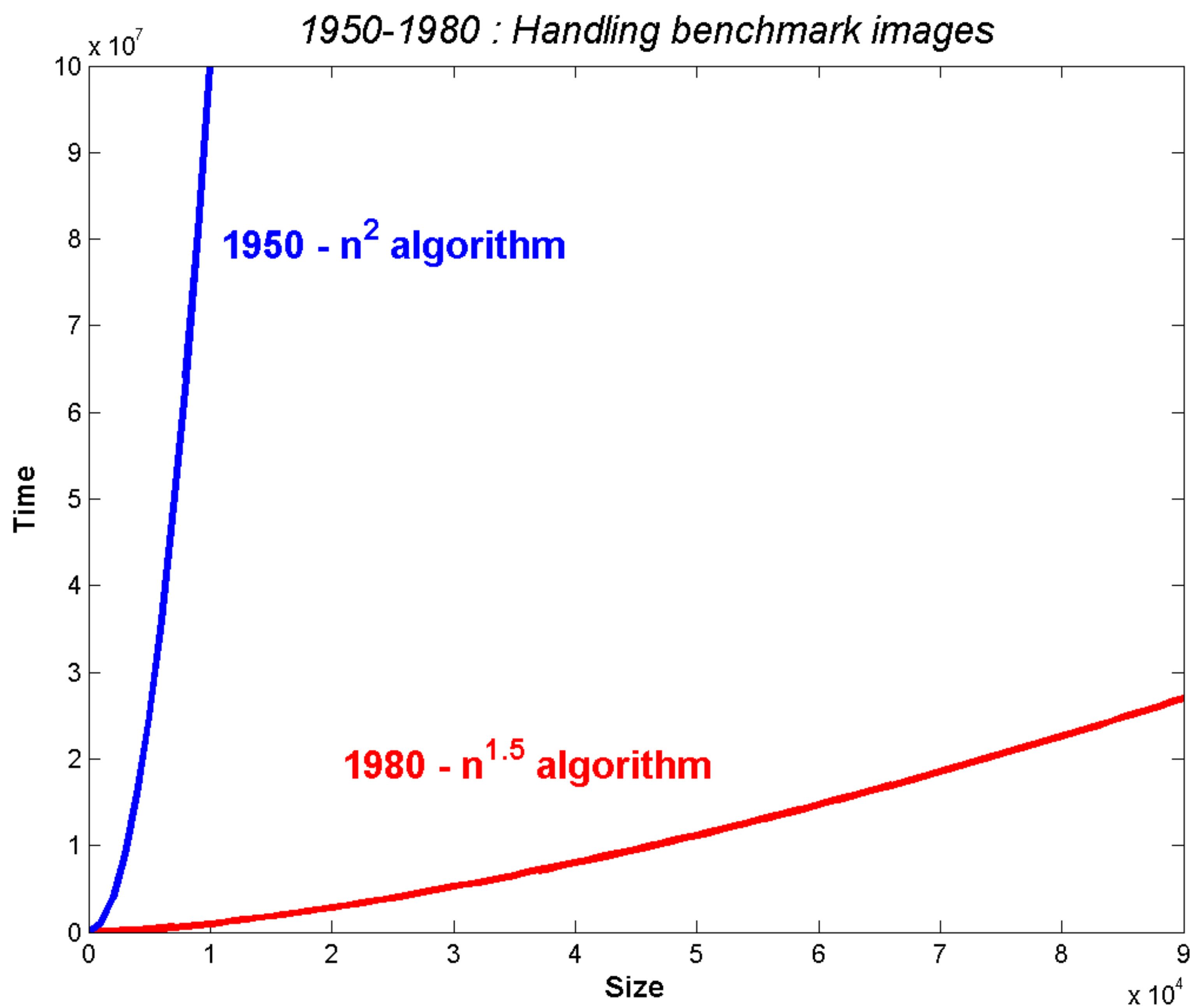
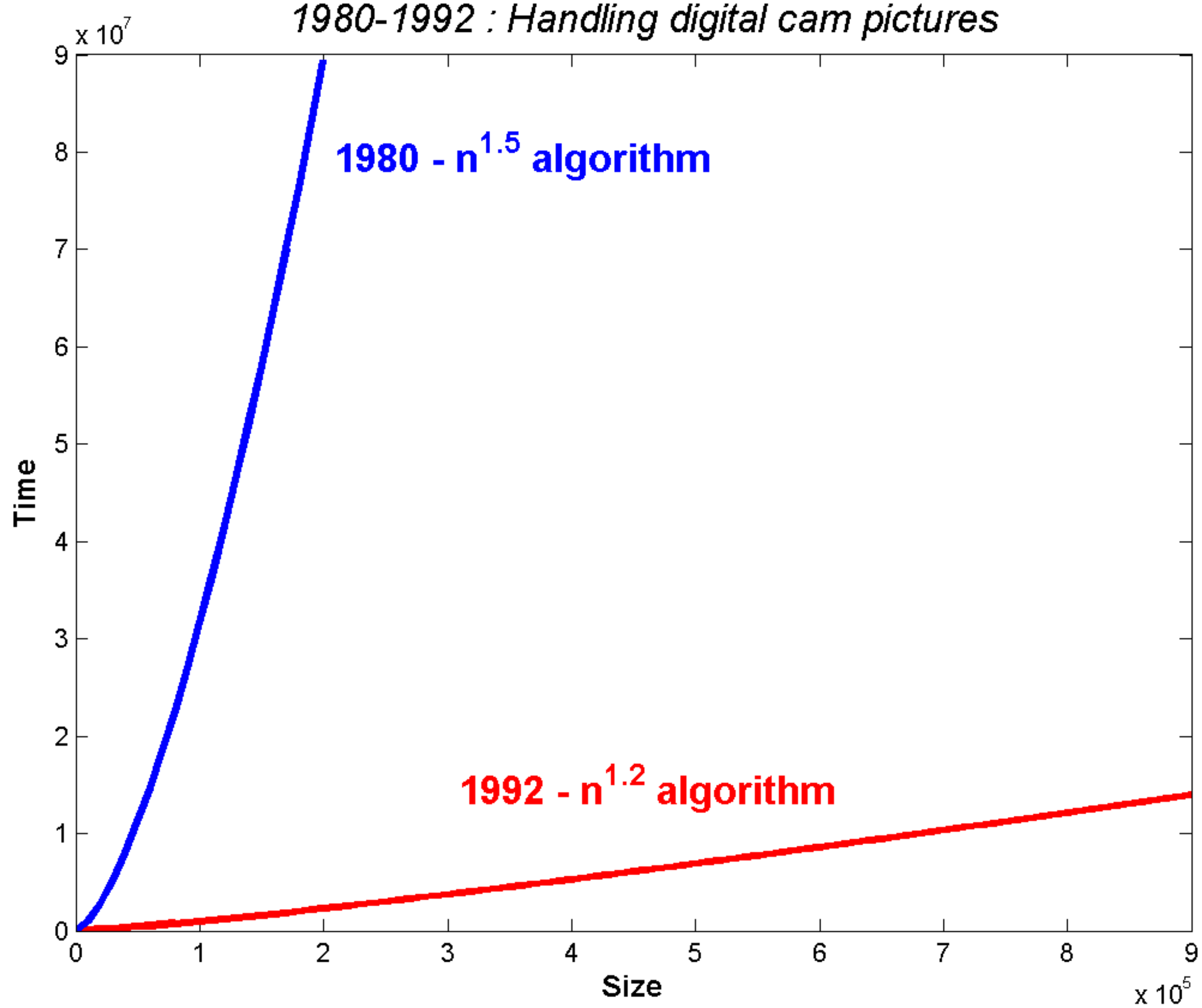


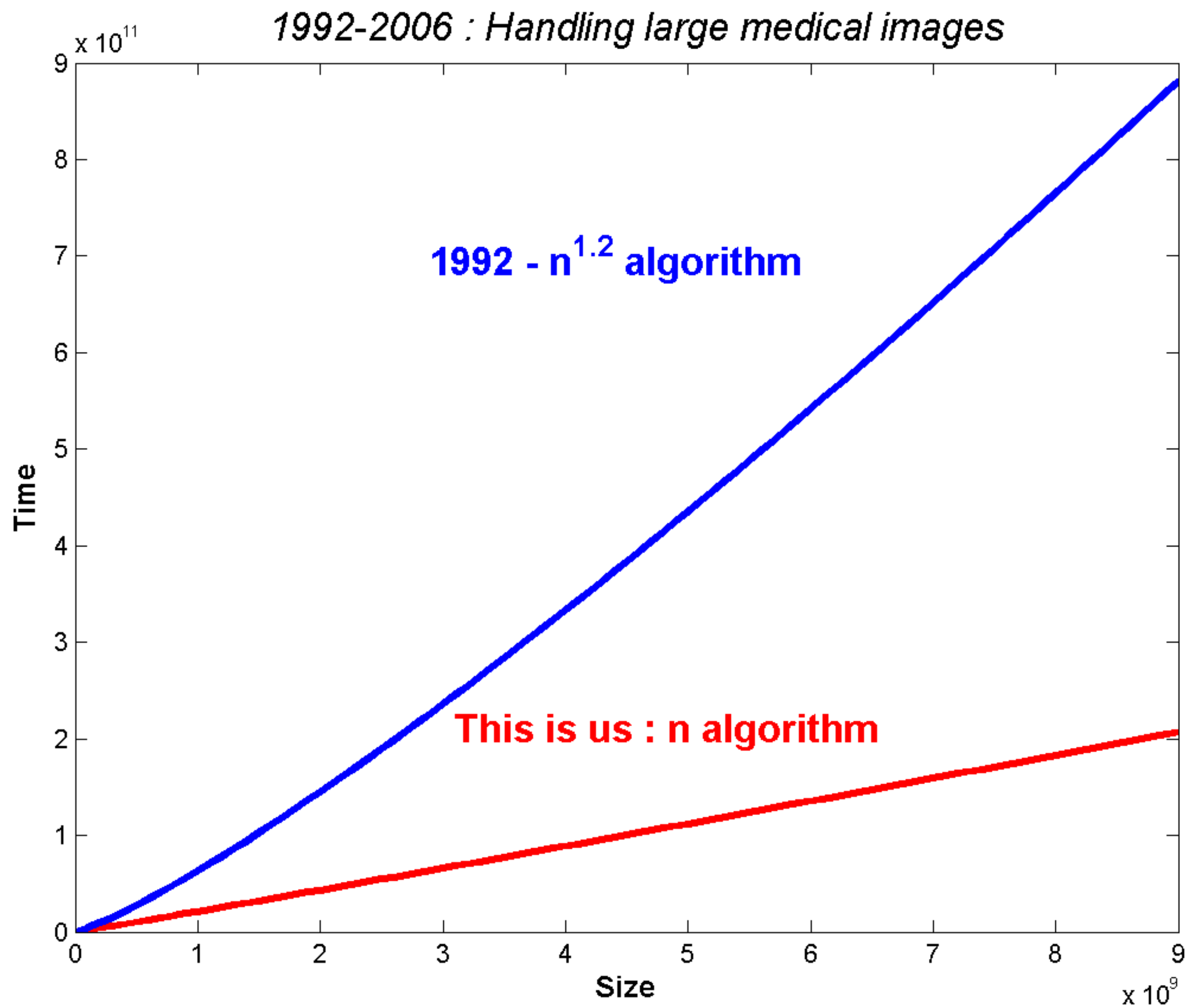
Image sizes up to 100K pixels

Dealing with larger images



Images up to 1 mega-pixel

Dealing with larger images



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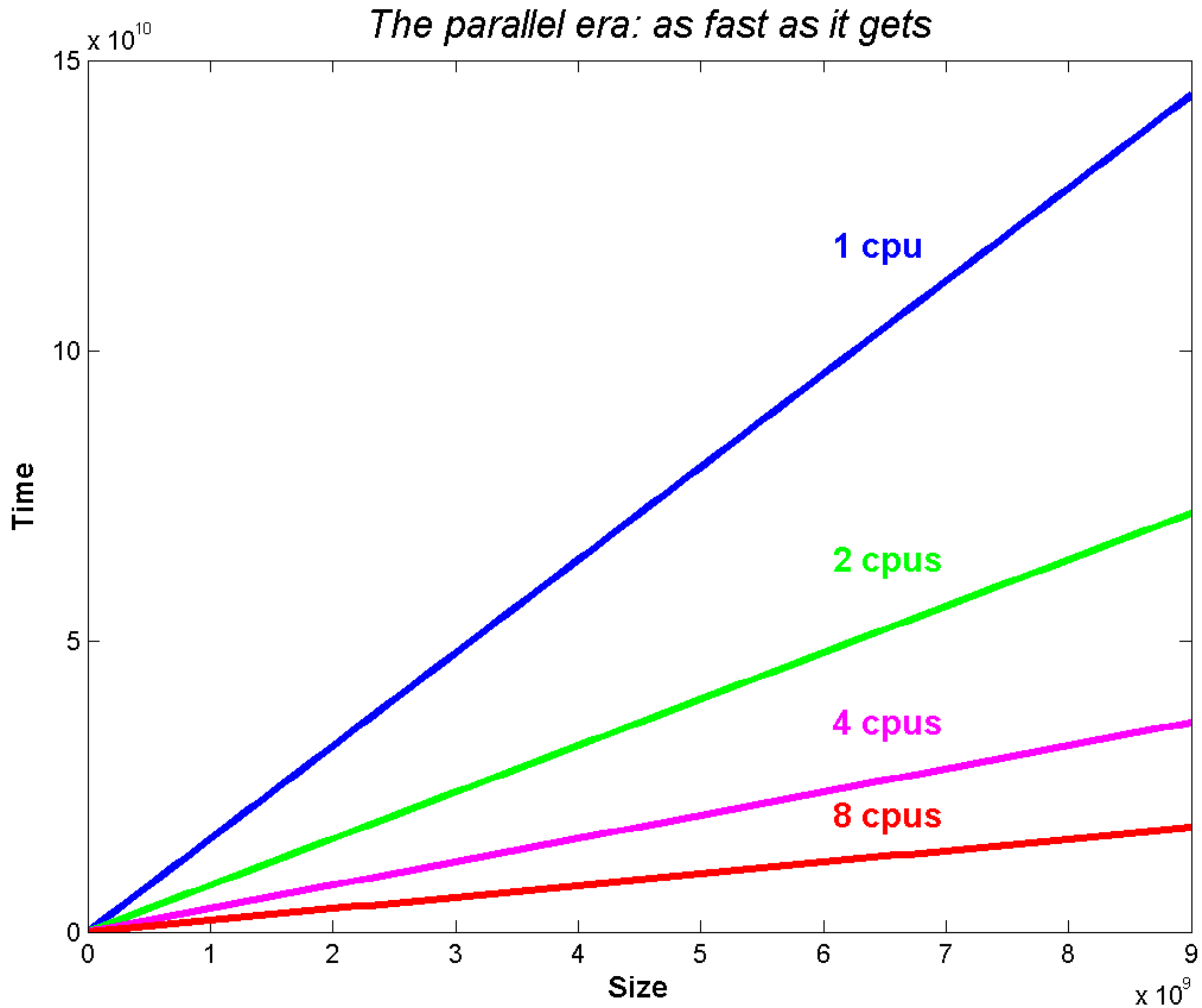
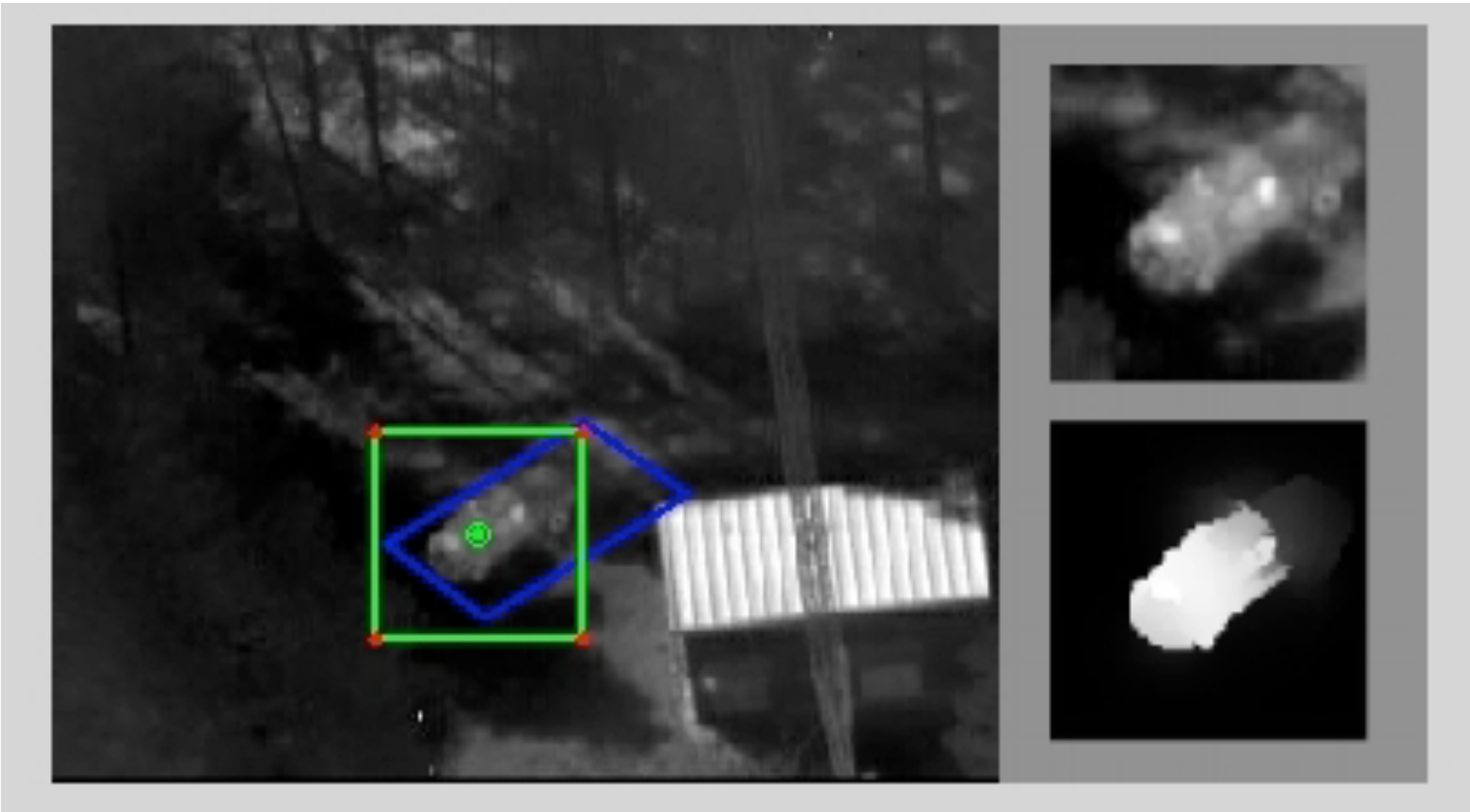
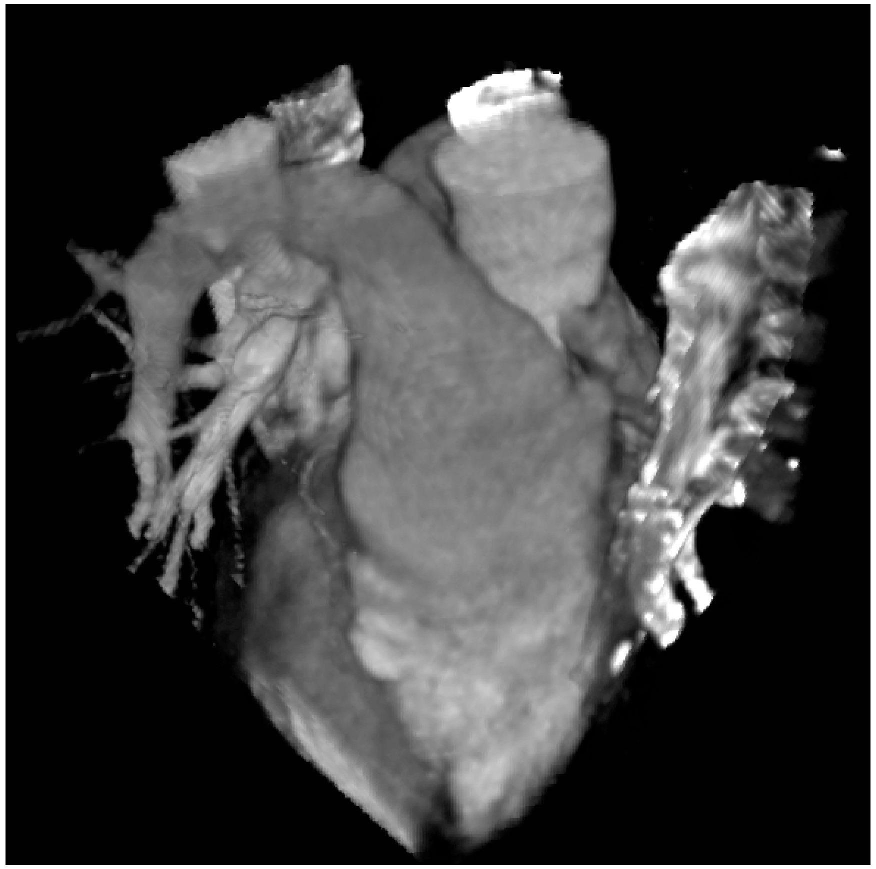


Giga-pixel images and beyond...

Image Segmentation in Surveillance



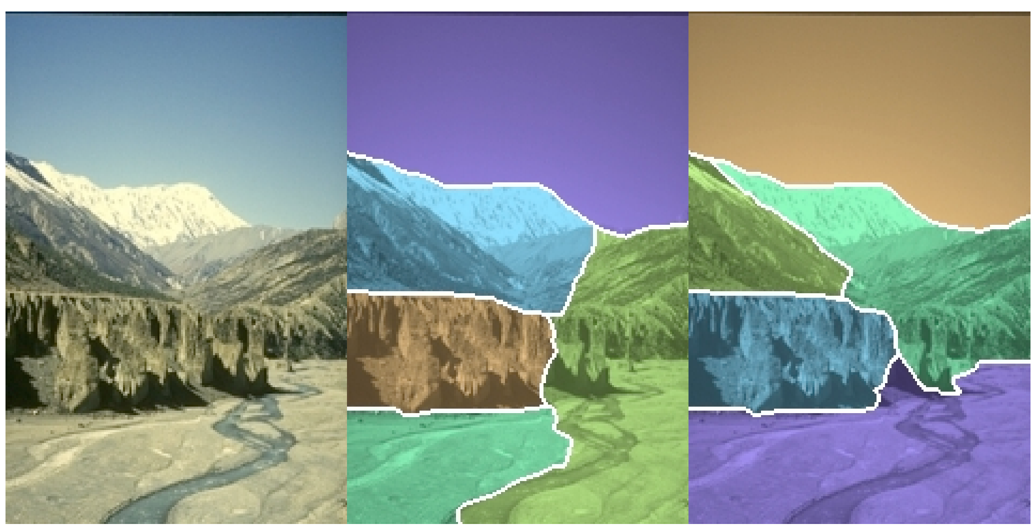
Thanks



Any image software can be improved by adding good image segmentation code.

$nc = .0093$ $nc = .0048$

$nc = .0085$ $nc = .0080$



$nc = .0081$ $nc = .0055$

$nc = .0047$ $nc = .0006$

Major Types of Image Segmentation

- Assisted Segmentation
  - Input from the consumer
  - Prior Knowledge (e.g. model of the heart)
- Unassisted Segmentation
  - No Prior Knowledge - No User Input

This Talk addresses a harder problem!

- Unassisted Segmentation without prior knowledge of the scene (image contents)

- Our methods can be used with prior information as well.

Image Segmentation

Probabilistic:

Besag `74
Geman & Geman `84
Veskler, Zabih, Boykov '98
..., Zhu, ... `01+
Tu et al. '05

Graph Partitioning:

Veskler, Zabih, Boykov '97+
Freeman & Perona '97
Shi & Malik '98+
Yu & Shi '03
Sharon et al. `00

Statistical:

Diday & Simon `80
Comaniciu & Meer `99+

Variational/Contour:

Kass, Witkin, Terzopoulos `88
Mumford & Shah `89+
Sethian '96+
Zhu, Lee, Yuille '95